A MODIFIED Y-M ACTION WITH THREE FAMILIES OF FERMIONIC SOLITONS AND PERTURBATIVE CONFINEMENT

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ABSTRACT

The dynamics of a four dimensional generally covariant modified SU(N) Yang-Mills action, which depends on the complex structure of spacetime and not its metric, is studied. A general solution of the complex structure integrability conditions is found in the context of the $G_{2,2}$ Grassmannian manifold, which admits a global SL(4,C) symmetry group. A convenient definition of the physical energy and momentum permits the study of the vacuum and soliton sectors. The model has a set of conformally SU(2,2) invariant vacua and a set of Poincaré invariant vacua. An algebraic integrability condition of the complex structure classifies the solitonic surfaces into three classes (families). The first class (spacetimes with two principal null directions) contains the Kerr-Newman complex structure, which has fermionic (electron-like) properties. That is the correct fermionic gyromagnetic ratio (g=2) and it satisfies the correct electron equations of motion. The conjugate complex structure determines the antisoliton, which has the same mass and opposite charge. The fermionic solitons are differentiated from the complex structure bosonic modes by the periodicity condition on compactified spacetime. The nonperiodicity of the found solitonic complex structures is proved. The modification of the Yang-Mills action has an essential consequence to the classical potential. It generates a linear static potential instead of the Coulomb-like $\frac{1}{r}$ potential of the ordinary Yang-Mills action. This linear potential implies that for every pure geometric soliton there are N solitonic gauge field excitations, which are perturbatively confined. The present model advocates a solitonic unification scheme without supersymmetry and/or superstrings.

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1 INTRODUCTION

Standard Model appears to be amazingly successful in all its experimental testings. These successes can be found in many recent books in Quantum Field Theory. But it is generally believed that it is not a complete theory because it contains too many independent parameters. On the other hand many apparent phenomena have not yet been proven or successfully described. Quark confinement, the three generations of leptons, the corresponding three generations of quarks and the apparent correspondence between leptons and quarks are some characteristic physical phenomena, which have not yet been understood in the context of Quantum Field Theory. These characteristic features are proved to occur in the present slightly modified generally covariant Yang-Mills model, which has fermionic solitons without fermionic fields in its action.

General Relativity is actually a well established macroscopic theory. It is based on the Einstein equation

$$E^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R \, g^{\mu\nu} = 8\pi k \, T^{\mu\nu} \qquad (1.1)$$

where $T^{\mu\nu}$ is the energy momentum tensor of the matter fields. $T^{\mu\nu}$ is an external non-geometric quantity, which is formally imposed by hand. The classical mathematical problem of this theory is to compute the metric tensor $g^{\mu\nu}$, which satisfies the Einstein equation. The external character of $T^{\mu\nu}$ has always been considered as a drawback of the theory and many efforts have been undertaken to derive it from geometry.

The two successful mainstreams (Quantum Field Theory and General Relativity) have been developed independently. Each branch has tried to incorporate the other one without apparent success. The straightforward "covariantization" of the Standard Model action with the Einstein gravitational term is not renormalizable. That is, it is not a self consistent Quantum Field Theory. These failures led researchers to look for non-conventional Lagrangian models. The mainstream of research turned first into Supergravity without success and after into Superstrings, without apparent experimental tests up to now. The first tests on supersymmetry are expected to be provided by the Large Hadron Collider (LHC) experiments. If supersymmetry is not found then research has to turn to more conventional models like the present solitonic one.

General Relativity researchers tried to generate particles in the context of Geometrodynamics, where matter is considered as a manifestation of geometry. The fundamental idea and expectation is to derive all particles from pure geometric quantities. The Einstein equation (1.1) is seen as the definition of the energy-momentum tensor of these particles. The Einstein-Infeld-Hoffman theory[3] of motion in General Relativity may be considered the origin of the geometrodynamic ideas. In this context the particle appears as a singularity of the Einstein tensor $E^{\mu\nu}$ and the equation of motion is derived from the self-consistency identity $E^{\mu\nu};\nu \equiv 0$ and the definitions of center of mass and the momenta. This result came up as a surprise for many researchers, who were used to the linear character of the Maxwell equations, and the problem has been

extensively studied[9]. Apparently the geometrodynamic point of view has also to postulate the (fundamental) equations, which the solitonic manifold must satisfy, and how the well known form of the Maxwell energy-momentum tensor is derived. In the Misner-Wheeler model[10] the fundamental equations are the Rainich conditions[27]. The metric configuration is the "particle" itself, and the electric charge is defined using an Einstein-Rosen wormhole[4] structure of space time. Despite its great philosophical appeal this program fails to describe many characteristic physical phenomena. It cannot explain the weak and the strong interactions, its solitonic particles have continuous mass and charge parameters, it is full of spinless solitons which do not appear in nature, etc, etc. One may think that the Rainich conditions are responsible for these unphysical consequences. But it has been proven that it is not the case. The Finkelstein-Misner topological analysis[6] showed that any set of (fundamental) equations applied to the metric tensor cannot imply soliton (particle, mass, etc.) discretization. It was this no-go result, that blocked any investigation for a new realistic geometrodynamic model and made this interesting idea to fade away.

Geometrodynamics has also failed to generate quantum phenomena but this effort had two extraordinary results which are purely microscopic. The first result is the derivation of the electron equations of motion[15],[2],[13] with the right terms without any effort of model building. The second one is the observation[1],[11] that the Kerr-Newman spacetime has the electron gyromagnetic ratio $g = 2$. These two results strongly suggested the identification of the electron with the charged Kerr manifold but the appropriate Quantum Field Theoretic model was missing. The value of the present model is that it may play this role or it may show the way how to find such a theory, which could incorporate these extraordinary results into Quantum Field Theory. This unification procedure does not need supersymmetry, because the fermionic particles appear as solitons of the model. That is the proposed particle unification scheme is of solitonic and not supersymmetric origin. In a solitonic unification schemes Standard Model is simply an effective action like the phonon actions in solids and fluids[28]. If the next few years the LHC experiments do not find supersymmetric particles, we have to turn to solitonic unification schemes as suggested by the present model.

The model started[21] as a simple exercise to find a four dimensional generally covariant action which would depend on the complex structure of the spacetime and not on its metric. Recall that this property characterizes the two dimensional string action. The purpose of this search was to find a renormalizable generally covariant action without higher order derivatives. Metric independence assures renormalizability[26], because the regularization procedure cannot generate non-renormalizable geometric terms. Only topological anomalies may appear. Calculations[26] of the first order one-loop diagrams in a convenient gauge condition show that they are finite. The action[22] of the model is reviewed in section 2, where the properties of the Lorentzian complex structure[7] are reviewed[8]. The new result of this section is the general solution of the complex structure integrability conditions using structure coordinates. A large part of the present paper is devoted to review the mathematical

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background because it is no used in current particle physics.

In section 3, the formalism of the Grassmannian manifolds and the classical domains is applied to reveal the invariance of the complex structures under the four dimensional global $SL(4, C)$ which is analogous to the $SL(2, C)$ symmetry of the string action. This mathematical background is necessary for the reader to understand the vacua and soliton sectors of the model and how global $SL(4, C)$ breaks down to the conformal $SU(2, 2)$ and the physical Poincaré symmetries, which are studied in section 4. The natural emergence of the Poincaré group is the most interesting result of the present model. It permits to find massive and massless stationary axisymmetric solitons and to classify the complex structures using the Hopf invariant. This rich physical content of the model is revealed in section 4. The complex structures with solitonic properties are classified[25] into three classes relative to the number of sheets of the complex structure. The first two-valued class is extensively studied. They are solitons because their complex structures cannot be compactified. The antisolitons are simply the complex conjugate complex structures of the solitons. Solitons and antisolitons have the same mass but opposite charges. The general forms of these massive and massless stationary axisymmetric solitons are computed. An analogous calculation indicates that the other two classes of solitons (with three and four sheets) do not contain stable massive solitons.

The model contains only a Yang-Mills field and the ordinary (null) tetrad which determines the Lorentzian complex structure. The symmetries of the model do not permit the existence of fermionic fields. In section 5 we show that the modification of the Yang-Mills action, which makes it independent of the metric tensor, has a characteristic physical consequence. The static potential of a source is no longer $\frac{1}{r}$ but it is linear, which could confine the "colored" sources[25]. From the two dimensional solitonic models we know that the solitons may be excited by the field modes. Analogous excitations are expected in the present case too. That is, the solitonic complex structures of the model may be excited by the gauge field modes. Then these excited solitons are perturbatively confined because of the linear gauge field potential. Only "colorless" states may exist free, which is a characteristic property of the hadrons. Notice that this confinement mechanism implies a strict correspondence between "leptonic" pure geometric solitons with vanishing gauge field and the "hadronic" ones with non-vanishing gauge field.

2   ACTION OF THE MODEL

The ordinary (Euclidean) almost complex structure is a <u>real</u> tensor $J_\mu^{\ \nu}$, normalized by the condition

$$J_\mu^{\ \rho} J_\rho^{\ \nu} = -\delta_\mu^\nu \tag{2.1}$$

It defines an (integrable) complex structure, if it satisfies the Nijenhuis integrability condition

$$J_\mu^{\ \sigma}\left(\partial_\sigma J_\rho^{\ \nu} - \partial_\rho J_\sigma^{\ \nu}\right) - J_\rho^{\ \sigma}\left(\partial_\sigma J_\mu^{\ \nu} - \partial_\mu J_\sigma^{\ \nu}\right) = 0 \tag{2.2}$$

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Then the manifold over which $J_\mu^{\ \nu}$ exists, becomes a complex manifold.

A complex structure is compatible with the metric tensor $g_{\mu\nu}$ of the manifold, if the two tensors satisfy the relation

$$J_\rho^{\ \mu} J_\sigma^{\ \nu} g_{\mu\nu} = g_{\rho\sigma} \qquad (2.3)$$

at any point of the manifold. If the signature of spacetime is Lorentzian, there is always a coordinate transformation such that the metric tensor takes the form of the Minkowski metric $\eta_{\mu\nu}$ at a given point. Then we see that the real tensor $J_\mu^{\ \nu}$ defines a Lorentz transformation at the given point. However there is no real Lorentz transformation, which satisfies the normalization condition (2.1) of the complex structure. Notice that this incompatibility is a pure local property and it is not related to the global structure of spacetime.

Hence the Lorentzian signature of spacetime is not compatible with a <u>real</u> tensor (complex structure) $J_\mu^{\ \nu}$. The notion of the Lorentzian complex structure has been generalized[7] to include complex tensors $J_\mu^{\ \nu}$. I anticipate that the existence of antisolitons in the present model is based on this particular property of the Lorentzian complex structure. This (modified) complex structure has been extensively studied by Flaherty[8]. It can be shown that there is always a null tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ such that the metric tensor and the complex structure tensor take the form

$$g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu$$

$$J_\mu^{\ \nu} = i(\ell_\mu n^\nu - n_\mu \ell^\nu - m_\mu \overline{m}^\nu + \overline{m}_\mu m^\nu) \qquad (2.4)$$

The integrability condition of this complex structure implies the Frobenius integrability conditions of the pairs (ℓ_μ, m_μ) and $(n_\mu, \overline{m}_\mu)$. That is

$$(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) = 0 \quad , \quad (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) = 0$$

$$(n^\mu m^\nu - n^\nu m^\mu)(\partial_\mu n_\nu) = 0 \quad , \quad (n^\mu m^\nu - n^\nu m^\mu)(\partial_\mu m_\nu) = 0 \qquad (2.5)$$

Frobenius theorem states that there are four complex functions $(z^\alpha, \ z^{\widetilde{\alpha}})$, $\alpha = 0, \ 1$, such that

$$dz^\alpha = f_\alpha \ \ell_\mu dx^\mu + h_\alpha \ m_\mu dx^\mu \quad , \quad dz^{\widetilde{\alpha}} = f_{\widetilde{\alpha}} \ n_\mu dx^\mu + h_{\widetilde{\alpha}} \ \overline{m}_\mu dx^\mu \qquad (2.6)$$

These four functions are the structure coordinates of the (integrable) complex structure. Notice that in the present case of Lorentzian spacetimes the coordinates $z^{\widetilde{\alpha}}$ are not complex conjugate of z^α, because $J_\mu^{\ \nu}$ is no longer a real tensor.

The reality conditions of the Newman-Penrose null tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ imply

$$dz^0 \wedge dz^1 \wedge d\overline{z^0} \wedge d\overline{z^1} = 0$$

$$dz^{\widetilde{0}} \wedge dz^{\widetilde{0}} \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \tag{2.7}$$

$$dz^{\widetilde{0}} \wedge dz^{\widetilde{0}} \wedge d\overline{z^{\widetilde{0}}} \wedge d\overline{z^{\widetilde{0}}} = 0$$

These relations are directly proven after a substitution of (2.6). They are equivalent to the existence of two real functions Ω_0, $\Omega_{\widetilde{0}}$ and a complex one Ω, such that

$$\Omega_0 \left(z^\alpha, \overline{z^\alpha} \right) = 0 \quad , \quad \Omega \left(z^{\widetilde{\alpha}}, \overline{z^\alpha} \right) = 0 \quad , \quad \Omega_{\widetilde{0}} \left(z^{\widetilde{\alpha}}, \overline{z^{\widetilde{\alpha}}} \right) = 0 \tag{2.8}$$

Notice that these relations provide an algebraic solution to the problem of complex structures on a spacetime. They are much easier handled than the PDEs (2.5). In the next section they will be transcribed in the $G_{2,2}$ Grassmannian manifold context providing a powerful mathematical machinery for the computation of complex structures.

The integrability conditions of the complex structure can be formulated in the spinor formalism. They imply that both spinors o^A and ι^A of the dyad satisfy the same PDE

$$\xi^A \xi^B \nabla_{AA'} \, \xi_B = 0 \tag{2.9}$$

where $\nabla_{AA'}$ is the covariant derivative connected to the vierbein $e_a{}^\mu$. This relation is equivalent to the existence of a complex vector field $\tau^{A'B}$ such that

$$\nabla^{A'}_{(A'} \, \xi_{B)} = \tau^{A'}_{(A} \xi_{B)} \tag{2.10}$$

Using the relation[18]

$$\nabla_{A'(A} \nabla^{A'}_B \, \xi_{C)} = \Psi_{ABCD} \xi^D \tag{2.11}$$

one can show that both o^A and ι^A satisfy the algebraic integrability condition

$$\Psi_{ABCD} \xi^A \xi^B \xi^C \xi^D = 0 \tag{2.12}$$

Namely, they are principal directions of the Weyl spinor Ψ_{ABCD}. Therefore a curved spacetime may admit a limited number of complex structures, which are directly related to its principal null directions. If the Weyl curvature vanishes, there is no restriction on the proper spinor basis. In this case the manifold is conformally flat and the integrability conditions are completely solved via Kerr's theorem[8].

2.1 Tetrad and structure coordinate forms of the action

The string action describes the dynamics of 2-dimensional surfaces in a multi-dimensional space. Its form

$$I_S = \frac{1}{2} \int d^2\xi \ \sqrt{-\gamma} \ \gamma^{\alpha\beta} \ \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{2.13}$$

does not essentially depend on the metric $\gamma^{\alpha\beta}$ of the 2-dimensional surface. It depends on its structure coordinates $(z^0, \ z^{\widetilde{0}})$, because in these coordinates it takes the metric independent form

$$I_S = \int d^2z \ \partial_0 X^\mu \partial_{\widetilde{0}} X^\nu \eta_{\mu\nu} \tag{2.14}$$

All the wonderful properties of the string model are essentially based on this characteristic feature of the string action.

The plausible question[21] and exercise is "what 4-dimensional action with first order derivatives depends on the complex structure but it does not depend on the metric of the spacetime?". The additional expectation is that such an action may be formally renormalizable because the regularization procedure will not generate geometric counterterms. The term "formally" is used because the 4-dimensional action may have anomalies which could destroy renormalizability, as it happens in the string action. Recall that the string and superstring actions are self-consistent only in precise dimensions, where the cancellation of the anomaly occurs.

A four dimensional action which satisfies the above criterion was found. The null tetrad form of this action[22] of the present model is

$$I_G = \int d^4x \ \sqrt{-g} \ \{(\ell^\mu m^\rho F_{j\mu\rho})(n^\nu \overline{m}^\sigma F_{j\nu\sigma}) + (\ell^\mu \overline{m}^\rho F_{j\mu\rho})(n^\nu m^\sigma F_{j\nu\sigma})\}$$

$$F_{j\mu\nu} = \partial_\mu A_{j\nu} - \partial_\nu A_{j\mu} - \gamma \, f_{jik} A_{i\mu} A_{k\nu}$$

$$\tag{2.15}$$

where $A_{j\mu}$ is a gauge field and $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ is an integrable null tetrad. The difference between the present action and the ordinary Yang-Mills action becomes more clear in the following form of the action.

$$I_G = -\frac{1}{8} \int d^4x \ \sqrt{-g} \ \left(2g^{\mu\nu} g^{\rho\sigma} - J^{\mu\nu} J^{\rho\sigma} - \overline{J^{\mu\nu}} \ \overline{J^{\rho\sigma}}\right) F_{j\mu\rho} F_{j\nu\sigma} \tag{2.16}$$

where $g_{\mu\nu}$ is a metric derived from the null tetrad and $J_\mu{}^\nu$ is the tensor of the integrable complex structure.

Like the 2-dimensional string action, the metric independence of the present action appears when we transcribe it in its structure coordinates form

$$I_G = \int d^4z \ F_{j01} F_{j\widetilde{0}\widetilde{1}} + comp. \ conj.$$

$$F_{j_{ab}} = \partial_a A_{jb} - \partial_a A_{jb} - \gamma \, f_{jik} A_{ia} A_{kb}$$

$$\tag{2.17}$$

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This transcription is possible because the metric and the integrable null tetrad take simple forms in the structure coordinates system.

In the case of the string action we do not need additional conditions because any orientable 2-dimensional surface admits a complex structure. But in the case of 4-dimensional surfaces, the integrability of the complex structure has to be imposed through precise conditions. These integrability conditions may be imposed either on the tetrad (2.5) or on the structure coordinates (2.7), using the ordinary procedure of Lagrange multipliers. These different possibilities will provide the various forms of the action which are equivalent, at least in the classical level. Its different variations should be seen as different ways to write down the integration measure over the complex structures of the 4-dimensional Lorentzian manifolds. The additional action term with the integrability conditions on the null tetrad is

$$I_C = -\int d^4x \ \{\phi_0(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu)+$$

$$+\phi_1(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) + \phi_{\widetilde{0}}(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu n_\nu)+ \qquad (2.18)$$

$$+\phi_{\widetilde{1}}(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu \overline{m}_\nu) + c.conj.\}$$

This Lagrange multiplier makes the complete action $I = I_G + I_C$ self-consistent and the usual quantization techniques may be used[24].

The local symmetries of the action are a) the well known local gauge transformations, b) the reparametrization symmetry as it is the case in any generally covariant action and c) the following extended Weyl transformation of the tetrad

$$\ell'_\mu = \chi_1 \ell_\mu \quad , \quad n'_\mu = \chi_2 n_\mu \quad , \quad m'_\mu = \chi m_\mu$$

$$\phi'_0 = \phi_0 \frac{\chi_2 \overline{\chi}}{\chi_1} \quad , \quad \phi'_1 = \phi_1 \frac{\chi_2 \overline{\chi}}{\chi}$$

$$\phi'_{\widetilde{0}} = \phi_{\widetilde{0}} \frac{\chi_1 \chi}{\chi_2} \quad , \quad \phi'_{\widetilde{1}} = \phi_{\widetilde{1}} \frac{\chi_1 \chi}{\overline{\chi}} \qquad (2.19)$$

$$g' = g(\chi_1 \chi_2 \chi \overline{\chi})^2$$

where χ_1, χ_2 are real functions and χ is a complex one.

2.2 Examples of complex structures

In order to make a selfconsistent paper we will present here some examples of complex structures, which can also be found in the works of Flaherty. The configurations of these complex structures will be used in the next sections.

The spinorial form of the integrability condition of the complex structure is conformally invariant. It is invariant under a spinor ξ^A multiplication with an arbitrary function, therefore we do not loose generality assuming the form $\xi^A = [1, \ \lambda]$. Then, in the Cartesian coordinates of a conformally flat spacetime

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the spinorial integrability conditions become the Kerr differential equations

$$\lambda^A \lambda^B \nabla_{A'A} \lambda_B = 0 \quad \Longleftrightarrow$$

$$(\partial_{0'0}\lambda) + \lambda(\partial_{0'1}\lambda) = 0 \ \ and \ \ (\partial_{1'0}\lambda) + \lambda(\partial_{1'1}\lambda) = 0 \tag{2.20}$$

where the Penrose spinorial notation is used with

$$x^{A'A} = x^\mu \sigma_\mu^{A'A} = \begin{pmatrix} x^0 + x^3 & (x^1 + ix^2) \\ (x^1 - ix^2) & x^0 - x^3 \end{pmatrix}$$

$$x_{A'A} = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} \tag{2.21}$$

$$\partial_{A'A} = \frac{\partial}{\partial x^{A'A}} = \sigma_{A'A}^\mu \partial_\mu = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix}$$

Kerr's theorem states that a general solution of these equations is any function $\lambda(x^{A'B})$, which satisfies a relation of the form

$$K(\lambda, \ x_{0'0} + x_{0'1}\lambda, \ x_{1'0} + x_{1'1}\lambda) = 0 \tag{2.22}$$

where $K(\cdot, \ \cdot, \ \cdot)$ is an arbitrary function.

Notice that in a conformally flat spacetime, the two solutions λ_1 and λ_2, which determine the spinor dyad $o^A \propto (1, \ \lambda_1)$ and $\iota^A \propto (1, \ \lambda_2)$, completely decouple. A characteristic example of a Minkowski spacetime complex structure is given by the two solutions of the quadratic (Kerr) function

$$(x - iy)\lambda^2 + 2(z - ia)\lambda - (x + iy) = 0 \tag{2.23}$$

where the ordinary Cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ are used. This Kerr function is time independent and determines a static complex structure. The two solutions are

$$\lambda_{1,2} = \frac{-(z - ia) \pm \sqrt{\Delta}}{x - iy} \quad , \quad \Delta = x^2 + y^2 + z^2 - a^2 - 2iaz \tag{2.24}$$

The corresponding spinor basis (dyad) is

$$o^A = [1 \ , \ \frac{-(z - ia) + \sqrt{\Delta}}{x - iy}] \quad , \quad \iota^A = -\frac{x - iy}{2\sqrt{\Delta}}[1, \ \frac{-(z - ia) - \sqrt{\Delta}}{x - iy}] \tag{2.25}$$

The corresponding null tetrad is

$$L \propto \left[(1 + \lambda_1\overline{\lambda_1})dt - (\lambda_1 + \overline{\lambda_1})dx - i(\overline{\lambda_1} - \lambda_1)dy - (1 - \lambda_1\overline{\lambda_1})dz\right]$$

$$M \propto \left[(1 + \lambda_1\overline{\lambda_2})dt - (\lambda_1 + \overline{\lambda_2})dx - i(\overline{\lambda_2} - \lambda_1)dy - (1 - \lambda_1\overline{\lambda_2})dz\right] \tag{2.26}$$

$$N \propto \left[(1 + \lambda_2\overline{\lambda_2})dt - (\lambda_2 + \overline{\lambda_2})dx - i(\overline{\lambda_2} - \lambda_2)dy - (1 - \lambda_2\overline{\lambda_2})dz\right]$$

which is the "flatprint" null tetrad of the Kerr-Newman manifold. In the case of $a = 0$ it becomes the trivial "spherical" complex structure.

In the case of conformally flat spacetimes the structure coordinates z^α are two independent functions of $(\lambda_1, \ x_{0'0} + x_{0'1}\lambda_1, \ x_{1'0} + x_{1'1}\lambda_1)$ and the structure coordinates $z^{\widetilde{\alpha}}$ are respectively two independent functions of $(\lambda_2, \ x_{0'0} + x_{0'1}\lambda_2, \ x_{1'0}+x_{1'1}\lambda_2)$. It is convenient to use the following structure coordinates

$$z^0 = t - \sqrt{\Delta} - ia \quad , \quad z^1 = \frac{-(z-ia)+\sqrt{\Delta}}{x-iy}$$

$$z^{\widetilde{0}} = t + z + \frac{x^2+y^2}{z+\sqrt{\Delta}-ia} \quad , \quad z^{\widetilde{1}} = -\frac{x-iy}{z+\sqrt{\Delta}-ia} \tag{2.27}$$

Notice that this complex structure cannot be defined over the whole Minkowski spacetime, because it is singular when $o^A \propto \iota^A$, which occurs at

$$z = 0 \quad , \quad x^2 + y^2 = a^2 \tag{2.28}$$

We will see below that these points do not belong to the Grassmannian manifold. Using the Lindquist coordinates $(t, \ r, \ \theta, \ \varphi)$

$$x = (r\cos\varphi + a\sin\varphi)\sin\theta$$

$$y = (r\sin\varphi - a\cos\varphi)\sin\theta \tag{2.29}$$

$$z = r\cos\theta$$

the structure coordinates take the form

$$z^0 = t - r + ia\cos\theta - ia \quad , \quad z^1 = e^{i\varphi}\tan\frac{\theta}{2}$$

$$z^{\widetilde{0}} = t + r - ia\cos\theta + ia \quad , \quad z^{\widetilde{1}} = -\frac{r+ia}{r-ia}e^{-i\varphi}\tan\frac{\theta}{2} \tag{2.30}$$

The Minkowski spacetime null tetrad takes the form

$$L_\mu dx^\mu = dt - dr - a\sin^2\theta \ d\varphi$$

$$N_\mu dx^\mu = \frac{r^2+a^2}{2(r^2+a^2\cos^2\theta)}[dt + \frac{r^2+2a^2\cos^2\theta-a^2}{r^2+a^2}dr - a\sin^2\theta \ d\varphi]$$

$$M_\mu dx^\mu = \frac{-1}{\sqrt{2}(r+ia\cos\theta)}[-ia\sin\theta \ (dt - dr) + (r^2 + a^2\cos^2\theta)d\theta+$$
$$+i\sin\theta(r^2 + a^2)d\varphi] \tag{2.31}$$

A simple way[23][25] to find a curved space complex structure is the Kerr-Schild ansatz

$$\ell_\mu = L_\mu \quad , \quad m_\mu = M_\mu \quad , \quad n_\mu = N_\mu + f(x) \ L_\mu \tag{2.32}$$

where the null tetrad $(L_\mu, \ N_\mu, \ M_\mu, \ \overline{M}_\mu)$ determines an integrable flat complex structure. In the case of the static (2.31) complex structure, $(\ell_\mu, \ n_\mu, \ m_\mu, \ \overline{m}_\mu)$ is integrable for

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$$f = \frac{h(r)}{2(r^2 + a^2 \cos^2 \theta)} \tag{2.33}$$

where $h(r)$ is an arbitrary function. Notice that for $h(r) = -2mr + e^2$ the Kerr-Newman space-time is found. A set of structure coordinates of the curved complex structure, which are smooth deformations of the Minkowski complex structure, are

$$z^0 = t - r + ia\cos\theta - ia \quad , \quad z^1 = e^{i\varphi}\tan\frac{\theta}{2}$$

$$\widetilde{z^0} = t + r - ia\cos\theta + ia - 2f_1 \quad , \quad \widetilde{z^1} = -\frac{r+ia}{r-ia}\, e^{2iaf_2}\, e^{-i\varphi}\tan\frac{\theta}{2} \tag{2.34}$$

where the two new functions are

$$f_1(r) = \int \frac{h}{r^2 + a^2 + h}\, dr \quad , \quad f_2(r) = \int \frac{h}{(r^2 + a^2 + h)(r^2 + a^2)}\, dr \tag{2.35}$$

In the present model these configurations are seen as solitons. The complex structure $J_\mu{}^\rho$ describes a fermionic soliton with charge e and its complex conjugate $\overline{J_\mu{}^\rho}$ describes an antisoliton with charge $-e$.

3   THE $G_{2,2}$ GRASSMANNIAN MANIFOLD

The present work is heavily based on projective spaces and the classical domains, therefore a short review of the projective Grassmannian manifolds and the $SU(2,2)$ classical domain is needed. The projective space CP^3 is the set of non vanishing 4-d complex vector Z^m , $m = 0, 1, 2, 3$ with the equivalence relation $X^m \sim Y^m$ if there exists a non vanishing complex number c such that $X^m = cY^m$. Then the natural topology of C^4 induces a well defined topology in CP^3. The coordinates Z^m are called homogeneous coordinates and the three coordinates $y^I = [\frac{Z^1}{Z^0} , \frac{Z^2}{Z^0} , \frac{Z^3}{Z^0}]$ are called projective coordinates in the $Z^0 \neq 0$ coordinate neighborhood. Every two elements X^{m1} and X^{m2} of CP^3 determine a 2×2 matrix $r_{A'B}$ such that

$$X^{mi} = \begin{pmatrix} \lambda^{Ai} \\ -ir_{A'B}\lambda^{Bi} \end{pmatrix} \tag{3.1}$$

where they are written in the chiral representation. Penrose[16] has observed that a general solution of the Kerr theorem, which determines the geodetic and shear free congruences, take the simple form $K(Z^m) = 0$, where $K(Z^m)$ is a homogeneous function. In the case of a first degree polynomial $K(Z^m) = S_m Z^m$ with $S_m = [S_A, S^{B'}]$ we may define ω_A

$$\omega_A \lambda^A = S_m Z^m = (S_A - iS^{B'} r_{B'A})\lambda^A \tag{3.2}$$

After a straightforward calculation we find that $\omega_A \equiv \rho_A - i\tau^{B'} r_{B'A}$ satisfy the differential equation

$$\partial_{A'(B}\omega_{C)} = \frac{\partial \omega_C}{\partial r^{A'B}} + \frac{\partial \omega_B}{\partial r^{A'C}} = 0 \tag{3.3}$$

Penrose points out that the inverse is also true. The space of the solutions of this differential equation is CP^3. He called this differential equation "twistor equation" and the projective space CP^3 twistor space. One can easily show that if X^{mi} of CP^3 satisfy the relations

$$X^{i\dagger} E X^j = 0 \quad , \quad \forall \, i, \, j \tag{3.4}$$

with

$$E = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3.5}$$

the generally complex 2×2 matrix $r_{A'B}$ becomes Hermitian and it transforms as the Cartesian coordinates of the Minkowski spacetime under the Poincaré subgroup of the projective transformations $SL(4, C)$ of CP^3. Penrose tried to extract physical meaning from these relations. I will not continue on twistor mode of thinking, because I want to avoid confusion of the present conventional quantum field theoretic model with the Penrose twistor program. But many times in the present work we will use the twistor formalism and its spinor notation, because it is computationally very effective.

In the case of a second degree polynomial $K(Z^m) = S_{mn}Z^mZ^n$ with

$$S_{mn} = \begin{pmatrix} S_{AB} & S_A{}^{B'} \\ S^{A'}{}_B & S^{A'B'} \end{pmatrix} \quad where \quad S_{AB} = S_{BA} , \ S^{A'}{}_B = S_B{}^{A'} \qquad (3.6)$$

we define the spinor ω_{AB}

$$\omega_{AB}\lambda^A\lambda^B = S_{mn}Z^mZ^n = (S_{AB} - iS_A^{A'}r_{A'B} - iS_B^{B'}r_{B'A} - S^{A'B'}r_{A'A}r_{B'B})\lambda^A\lambda^B$$
$$(3.7)$$

which satisfies the relation

$$\partial_{A'(B}\omega_{CD)} = 0 \qquad (3.8)$$

In the case of a fourth degree polynomial $K(Z^m) = S_{mnpq}Z^mZ^nZ^pZ^q$ we find

$$\omega_{ABCD} = S_{ABCD} - iS^{A'}_{(ACD}r_{A'B)} - S^{A'B'}_{(CD}r_{A'A}r_{B'B)} +$$
$$+ iS^{A'B'C'}_{(D}r_{A'A}r_{B'B}r_{C'C)} + iS^{A'B'C'D'}r_{A'A}r_{B'B}r_{C'C}r_{C'C} \qquad (3.9)$$

which also satisfies the twistor equation

$$\partial_{A'(B}\omega_{CDEF)} = 0 \qquad (3.10)$$

It is proved that a spinor λ^A, which satisfies the fourth degree homogeneous polynomial

$$\omega_{ABCD}\lambda^A\lambda^B\lambda^C\lambda^D = 0 \qquad (3.11)$$

determines a geodetic and shear free congruence in Minkowski spacetime. Two roots of this polynomial define a complex structure. This relation is useful, because it will coincide with the algebraic integrability condition on the curved spacetime in the weak gravity approximation. That is, we expect the Weyl spinorial tensor Ψ_{ABCD} to become proportional with ω_{ABCD} in the weak gravity limit and in an appropriate (Cartesian) coordinate system.

Let us now turn to the definition of the Grassmannian projective manifold $G_{2,2}$. Consider the set of the 4×2 complex matrices of rank 2

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \qquad (3.12)$$

with the equivalence relation $T \sim T'$ if there exists a 2×2 regular matrix S such that

$$T' = TS \qquad (3.13)$$

The coordinates

$$z = T_2T_1^{-1} \qquad (3.14)$$

completely determine the points of the set. The topology of the 4×2 matrices implies a well defined topology in this projective manifold $G_{2,2}$. The coordinates T are called homogeneous coordinates and the coordinates z are called projective coordinates. Under a general linear 4×4 transformation

$$\begin{pmatrix} T_1' \\ T_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \qquad (3.15)$$

the inhomogeneous coordinates transform as

$$z' = (A_{21} + A_{22}\ z)\,(A_{11} + A_{12}\ z)^{-1} \tag{3.16}$$

which is called fractional transformation and it preserves the compact manifold $G_{2,2}$, which is called Grassmannian manifold.

3.1 Bounded and unbounded realizations of the SU(2,2) classical domain

The points of $G_{2,2}$ with positive definite 2×2 matrix

$$\begin{pmatrix} T_1^\dagger & T_2^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} > 0 \quad \Longleftrightarrow \quad I - z^\dagger z > 0 \tag{3.17}$$

is the bounded $SU(2,2)$ classical domain[19], because it is bounded in the general z-space and it is invariant under the $SU(2,2)$ transformation

$$\begin{pmatrix} T_1' \\ T_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$z' = (A_{21} + A_{22}\ z)\,(A_{11} + A_{12}\ z)^{-1}$$

$$A_{11}^\dagger A_{11} - A_{21}^\dagger A_{21} = I \quad , \quad A_{11}^\dagger A_{12} - A_{21}^\dagger A_{22} = 0 \quad , \quad A_{22}^\dagger A_{22} - A_{12}^\dagger A_{12} = I \tag{3.18}$$

The characteristic (Shilov) boundary of this domain is the $S^1 \times S^3 [= U(2)]$ manifold with $z^\dagger z = I$. The ordinary parametrization of this boundary is

$$U = e^{i\tau} \begin{pmatrix} \cos\rho + i\sin\rho\cos\theta & i\sin\rho\sin\theta\ e^{-i\varphi} \\ i\sin\rho\sin\theta\ e^{i\varphi} & \cos\rho - i\sin\rho\cos\theta \end{pmatrix} =$$

$$= \tfrac{1+r^2-t^2+2it}{[1+2(t^2+r^2)+(t^2-r^2)^2]} \begin{pmatrix} 1+t^2-r^2-2iz & -2i(x-iy) \\ -2i(x+iy) & 1+t^2-r^2+2iz \end{pmatrix} \tag{3.19}$$

where $\tau \in (-\pi, \pi)$, $\varphi \in (0, 2\pi)$, $\rho \in (0, \pi)$, $\theta \in (0, \pi)$.

In the homogeneous coordinates

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \tag{3.20}$$

we have

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \tag{3.21}$$

and the positive definite condition takes the form

$$\begin{pmatrix} H_1^\dagger & H_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} > 0 \quad \Longleftrightarrow \quad -i(r - r^\dagger) = y > 0 \tag{3.22}$$

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where the projective coordinates $r_{A'B} = x_{A'B} + iy_{A'B}$ are defined as $r = iH_2H_1^{-1}$, which implies $H_2 = -irH_1$ and

$$r = i(I+z)(I-z)^{-1} = i(I-z)^{-1}(I+z)$$

$$z = (r-iI)(r+iI)^{-1} = (r+iI)^{-1}(r-iI)$$

(3.23)

The fractional transformations which preserve the unbounded domain are

$$\begin{pmatrix} H_1' \\ H_2' \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

$$r' = (B_{22}\ r + iB_{21})(B_{11} - iB_{12}\ r)^{-1}$$

$$B_{11}^\dagger B_{22} + B_{21}^\dagger B_{12} = I \quad , \quad B_{11}^\dagger B_{21} + B_{21}^\dagger B_{11} = 0 \quad , \quad B_{22}^\dagger B_{12} + B_{12}^\dagger B_{22} = 0$$

(3.24)

The characteristic boundary in this "upper plane" realization of the classical domain is the "real axis"

$$y_{A'A} = 0 \tag{3.25}$$

Using the $SU(2,2)$ generators, the $SL(4,C)$ infinitesimal transformations have the form

$$\delta Y = \tfrac{i}{2}\epsilon^\mu \gamma_\mu (1+\gamma_5) Y \quad , \quad \delta Y = -\tfrac{i}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu} Y$$

$$\delta Y = -\tfrac{i}{2}\kappa^\mu \gamma_\mu (1-\gamma_5) Y \quad , \quad \delta Y = -\tfrac{1}{2}\rho\gamma_5 Y$$

(3.26)

The real parts of the infinitesimal variables (ϵ^μ, $\epsilon^{\mu\nu}$, κ^μ, ρ) provide the 15 $SU(2,2)$ charges and their imaginary parts the remaining 15 charges of the $SL(4,C)$ transformations.

Considering the explicit forms of the homogeneous coordinates we have

$$H = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} = \begin{pmatrix} \lambda^{01} & \lambda^{02} \\ \lambda^{11} & \lambda^{12} \\ -i(r_{0'0}\lambda^{01} + r_{0'1}\lambda^{11}) & -i(r_{0'0}\lambda^{02} + r_{0'1}\lambda^{12}) \\ -i(r_{1'0}\lambda^{01} + r_{1'1}\lambda^{11}) & -i(r_{1'0}\lambda^{02} + r_{1'1}\lambda^{12}) \end{pmatrix} \tag{3.27}$$

where everything has been arranged such that the spinor transformations imply the corresponding spacetime transformations and vice-versa.

If we restrict the above $z_{ij} \to r_{A'B}$ transformations at the Shilov boundary we find the form

$$t = \frac{\sin\tau}{\cos\tau - \cos\rho}$$

$$x + iy = \frac{\sin\rho}{\cos\tau - \cos\rho} \sin\theta\ e^{i\varphi} \tag{3.28}$$

$$z = \frac{\sin\rho}{\cos\tau - \cos\rho} \cos\theta$$

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Additional formulas are

$$r = \frac{\sin\rho}{\cos\tau - \cos\rho} = \frac{-\sin\rho}{2\sin\frac{\tau+\rho}{2}\sin\frac{\tau-\rho}{2}}$$

$$\sqrt{1 + 2(t^2 + r^2) + (t^2 - r^2)^2} = \frac{2}{\cos\tau - \cos\rho} \tag{3.29}$$

where now $r = \sqrt{x^2 + y^2 + z^2}$ denotes the ordinary radial component and takes positive values. Notice that the Cartesian coordinates (x, y, z) are the projective coordinates from the center of S^3. We essentially need two such tangent planes to cover the whole sphere (but equator). The two hemispheres are covered by permitting the radial variable r to take negative values too.

We also see that

$$t - r = -\cot\frac{\tau-\rho}{2}$$

$$t + r = -\cot\frac{\tau+\rho}{2} \tag{3.30}$$

Through the above transformation Minkowski spacetime is conformally equivalent to the half of $S^1 \times S^3$. It is easy to prove that the following two points of $S^1 \times S^3$ correspond to the same point (t, x, y, z) of the Minkowski space.

$$(\tau, \rho, \theta, \varphi) \Longrightarrow (t, x, y, z)$$

$$(\tau + \pi , \ \pi - \rho , \ \pi - \theta , \ \varphi + \pi) \Longrightarrow (t, x, y, z) \tag{3.31}$$

In the τ, ρ axes the $r = \infty$ boundaries are $\tau - \rho = 0$ and $\tau + \rho = 0$. These are the two Penrose boundaries $\mathfrak{J}^\pm$ of Minkowski spacetime. The two triangles at both sides of the ρ axis have $r > 0$ and the other two triangles have $r < 0$.

Minkowski spacetime may be "properly" compactified by simply identifying its two Penrose boundaries $\mathfrak{J}^+$ and $\mathfrak{J}^-$. This is naturally done through the identification of compactified Minkowski spacetime with the characteristic boundary of the type IV $SO(2,4)$ invariant classical domain, which is the part of CP^5 determined by the relations[19]

$$t^{\mathsf{T}} H t = 0 \quad , \quad t^\dagger H t > 0 \quad , \quad \mathrm{Im}\,\frac{t_1}{t_0} > 0 \tag{3.32}$$

where t is a 6-dimensional complex column (the homogeneous coordinates of CP^5) and $H = diag[1, 1, -1, -1, -1, -1]$. Because of the homomorphism between $SU(2,2)$ and $O(2,4)$, the homogeneous coordinates (X^{mi}) of $G_{2,2}$ are related to the homogeneous coordinates of CP^5 with the following relations[18]

$$t_0 = \frac{i}{\sqrt{2}}(R^{12} - R^{03}) \quad , \quad t_1 = R^{01} + \frac{1}{2}R^{23}$$

$$t_2 = R^{01} - \frac{1}{2}R^{23} \quad , \quad t_3 = \frac{i}{\sqrt{2}}(R^{02} - R^{13}) \tag{3.33}$$

$$t_4 = \frac{1}{\sqrt{2}}(R^{02} + R^{13}) \quad , \quad t_5 = \frac{-i}{\sqrt{2}}(R^{12} + R^{03})$$

where $R^{mn} = X^{m1}X^{n2} - X^{n1}X^{m2}$. Recall that the homomorphisms between the three conformal groups are

$$SU(2,2) \overset{2\to1}{===}> O^\uparrow_+(2,4) \overset{2\to1}{===}> C^\uparrow_+(1,3) \tag{3.34}$$

3.2 Complex structures in $G_{2,2}$ context

In the simple case of conformally flat spacetimes, the integrability condition of the complex structure can be solved by Kerr's theorem[7]. Using the $G_{2,2}$ homogeneous coordinates X^{mi} this general solution takes the form[18]

$$\overline{X^{mi}}E_{mn}X^{nj} = 0$$
$$\tag{3.35}$$
$$K_i(X^{mi}) = 0$$

where the first line relations fix the surface to the Shilov boundary or a part of it, and the second line relations are the two Kerr homogeneous function. The structure coordinates z^α are then two independent functions of $\frac{X^{m1}}{X^{01}}$ and $z^{\widetilde\alpha}$ are two independent functions of $\frac{X^{m2}}{X^{02}}$.

This particular solution indicates the form of a general solution for the integrability condition of the complex structure on a generally curved spacetime. In this case the $G_{2,2}$ homogeneous coordinates X^{mi} have to satisfy relations of the form

$$\Omega_{ij}(\overline{X^{mi}}, X^{nj}) = 0$$
$$\tag{3.36}$$
$$K_i(X^{mi}) = 0$$

where all the functions are homogeneous relative to X^{n1} and X^{n2} independently. That is, they are defined in $CP^3 \times CP^3$. The rank-2 condition on the matrix X^{mi} defines the solutions as $SL(2,C)$ fiber bundles on 4-dimensional surfaces of $G_{2,2}$. The structure coordinates $(z^\alpha, z^{\widetilde\alpha})$ are determined exactly like in the simple case of conformally flat spacetimes given above.

In General Relativity the asymptotic flatness condition is imposed using the metric. In the present case of complex structures this condition is imposed through the assumption that there are independent homogeneous transformations of X^{n1} and X^{n2} such that

$$\overline{X^{m1}}E_{mn}X^{n1} = 0$$
$$\overline{X^{m2}}E_{mn}X^{n2} = 0 \tag{3.37}$$
$$\overline{X^{m1}}E_{mn}X^{n2} \neq 0$$

That is the two functions $\Omega_{11}(\overline{X^{m1}}, X^{n1})$ and $\Omega_{22}(\overline{X^{m2}}, X^{n2})$ take the flat space forms. The first two annihilations will be used below to restrict the forms of stationary axisymmetric complex structures.

The central problem of the present work is to find solutions $X^{mi}(x)$ which satisfy relations of the form (3.36). The topological classes of these solutions will determine the soliton sectors of the model. The algebraic nature of the

two homogeneous Kerr functions $K_i(X^{mi})$ is a powerful mathematical property which will be used below. But from the physical point of view it is somehow obscure because it hides physical intuition. Therefore one way to replace them is the parametrization (3.1) of X^{mi} where λ^{Ai} are functions of $r^{A'A}$ which satisfy the Kerr differential equations

$$\lambda^A \lambda^B \frac{\partial}{\partial r^{A'A}} \lambda_B = 0 \qquad (3.38)$$

which was our intuitive procedure for the discovery of the form (3.36) of the general solution.

Another physically very intuitive form, which replaces the Kerr functions, is the following trajectory parametrization of X^{mi}

$$X^{mi} = \begin{pmatrix} \lambda^{Ai} \\ -i\xi^i_{A'B}(\tau_i)\lambda^{Bi} \end{pmatrix} \qquad (3.39)$$

where $\xi^i_{A'B}(\tau_i)$ are two complex trajectories in the Grassmannian manifold $G_{2,2}$. A combination of this parametrization with the Grassmannian one (3.1) implies the two conditions $\det[r_{A'B} - \xi^i_{A'B}(\tau_i)] = 0$ for the two linear equations $[r_{A'B} - \xi^i_{A'B}(\tau_i)]\lambda^{Bi} = 0$ to admit non-vanishing solutions. Notice that this condition (restricted to the Shilov boundary) is identical to the relation

$$\eta_{\mu\nu}(x^\mu - \xi^\mu(\tau))(x^\nu - \xi^\nu(\tau)) = 0 \qquad (3.40)$$

which was first used by Newman and coworkers[14] to determine twisted geodetic and shear free null congruences in Minkowski spacetime.

It is to prove that the quadratic Kerr polynomial

$$Z^1 Z^2 - Z^0 Z^3 + 2aZ^0 Z^1 = 0 \qquad (3.41)$$

is implied by the trajectory $\xi^\mu(\tau) = (\tau \ , \ 0 \ , \ 0 \ , \ ia)$.

In the case of one trajectory, we will have one Kerr function. In this case and for a trajectory normalized by $\xi^0(\tau) = \tau$, the asymptotic flatness condition implies

$$i(\overline{\tau} - \tau) - i(\overline{\xi^1} - \xi^1)\frac{\lambda + \overline{\lambda}}{1 + \lambda\overline{\lambda}} + (\overline{\xi^2} - \xi^2)\frac{\overline{\lambda} - \lambda}{1 + \lambda\overline{\lambda}} - i(\overline{\xi^3} - \xi^3)\frac{1 + \lambda\overline{\lambda}}{1 + \lambda\overline{\lambda}} = 0 \quad (3.42)$$

which fixes the imaginary parts of the two complex parameters τ_1 and τ_2.

Using the $G_{2,2}$ formalism, the (2.23) complex structure of Minkowski spacetime is implied by the quadratic homogeneous polynomial (3.41). Notice that the points of spacetime (2.28) with $\det(\lambda^{Ai}) = 0$ do not belong to the Grassmannian projective manifold, because the corresponding 4×2 matrix has not rank 2. This means that if the spacetime is defined as a surface of of $G_{2,2}$, these points do not belong to the spacetime!

Using the Kerr-Schild ansatz, we have derived a general curved complex structure (2.33). Assuming that the Kerr function of the curved complex

structure is the same with that of the corresponding flatprint complex structure we can find the 4-dimensional surface of $G_{2,2}$. Its explicit form for the Kerr-Newman complex structure $(h(r) = -2mr + q^2)$ in Lindquist coordinates (t, r, θ, φ) is

$$x^0 = t + \frac{f_1}{\cos^2 f + \cos^2 \theta \sin^2 f}$$

$$x^1 + ix^2 = \frac{(r+f_1)\cos f + a\sin^2 \theta \sin f}{\cos^2 f + \cos^2 \theta \sin^2 f} \sin \theta e^{i\varphi} e^{-if}$$

$$x^3 = r\cos \theta + \frac{f_1 \cos \theta}{\cos^2 f + \cos^2 \theta \sin^2 f} +$$
$$+ \frac{\sin f \sin^2 \theta}{\cos^2 f + \cos^2 \theta \sin^2 f}[(r\sin f - a\cos f)\cos \theta + f_1 \sin f]$$

$$y^0 = 0$$

$$y^1 + iy^2 = \frac{(r+f_1)\sin f - a\cos f}{\cos^2 f + \cos^2 \theta \sin^2 f} \cos \theta \sin \theta e^{i\varphi} e^{-if}$$

$$y^3 = \frac{\cos f \sin^2 \theta}{\cos^2 f + \cos^2 \theta \sin^2 f}[a\cos f - (r+f_1)\sin f]$$

(3.43)

where the two functions entering the configuration are

$$f = \frac{a}{2\sqrt{a^2+q^2-m^2}} \arctan \frac{2(r-m)\sqrt{a^2+q^2-m^2}}{r^2 - 2mr + +2m^2 - a^2 - q^2}$$

$$f_1 = -m \ln \frac{r^2 - 2mr + a^2 + q^2}{m^2} + \frac{2m^2 - q^2}{a} f$$

(3.44)

Notice that this surface is outside the classical domain because $y^0 = 0$.

As a second example we will present below the "natural" complex structure of the $U(2)$ surface. The 1-forms of the left invariant generators of the group $U(2)$ are defined by the relation $U^\dagger dU = ie_L^a \sigma_a = i(e_L^0 \sigma^0 - e_L^i \sigma^i)$. In the (3.19) parametrization the 1-forms are

$$e_L^0 = d\tau$$

$$e_L^1 = -\sin\theta\cos\varphi d\rho + (\sin^2\rho\sin\varphi - \sin\rho\cos\rho\cos\theta\cos\varphi)d\theta +$$
$$+ (\sin^2\rho\cos\theta\sin\theta\cos\varphi + \sin\rho\cos\rho\sin\theta\sin\varphi)d\varphi$$

$$e_L^2 = -\sin\theta\sin\varphi d\rho + (-\sin\rho\cos\rho\cos\theta\sin\varphi + \sin^2\rho\cos\varphi)d\theta +$$
$$+ (\sin^2\rho\cos\theta\sin\theta\sin\varphi - \sin\rho\cos\rho\sin\theta\cos\varphi)d\varphi$$

$$e_L^3 = -\cos\theta d\rho + \sin\rho\cos\rho\sin\theta d\theta - \sin^2\rho\sin^2\theta d\varphi$$

(3.45)

In Cartesian coordinates the generators take the form $[C = \frac{4}{1+2(t^2+r^2)+(t^2-r^2)^2}]$

$$e_L^0 = C[(1 + r^2 + t^2)dt - 2txdx - 2tydy - 2tzdz]$$

$$\begin{aligned}e_L^1 = C[&-2xtdt + (1 + t^2 + x^2 - y^2 - z^2)dx+ \\ &+2(xy + z)dy + 2(xz - y)dz]\end{aligned}$$

$$\begin{aligned}e_L^2 = C[&-2ytdt + 2(xy - z)dx + (1 + t^2 - x^2 + y^2 - z^2)dy+ \\ &+2(yz + x)dz]\end{aligned}$$
(3.46)

$$\begin{aligned}e_L^3 = C[&-2ztdt + 2(xz + y)dx + 2(yz - x)dy+ \\ &+(1 + t^2 - x^2 - y^2 + z^2)dz]\end{aligned}$$

The 1-forms satisfy the following differential relations

$$de_L^0 = 0 \quad , \quad de_L^i = \epsilon_{ijk}e_L^j \wedge e_L^k \tag{3.47}$$

which imply the relations

$$\left(e^i e^j \partial e^k\right) = e^{i\mu}e^{j\nu}\left(\partial_\mu e_\nu^k - \partial_\nu e_\mu^k\right) = 2\epsilon_{ijk} \tag{3.48}$$

The "natural" complex structure on $S^1 \times S^3$ is defined by the following tetrad

$$L^\mu = e_L^{0\mu} - e_L^{3\mu}$$

$$N^\mu = e_L^{0\mu} - e_L^{3\mu} \tag{3.49}$$

$$M^\mu = e_L^{1\mu} + ie_L^{2\mu}$$

This complex structure is generated from the following degenerate quadratic polynomial, which is the product of two linear polynomials

$$(Z^1 + Z^3)(Z^0 + Z^2) = 0 \tag{3.50}$$

The surface is the boundary of the classical domain $y_{A'A} = 0$ with the following homogeneous coordinates of $G_{2,2}$

$$H = \begin{pmatrix} 1 & \frac{x-iy}{t-z+i} \\ \frac{x+iy}{t+z+i} & 1 \\ -i\left[t - z - (x - iy)\lambda_1\right] & \frac{-i}{\lambda_2}\left[t - z - (x - iy)\lambda_2\right] \\ -i\left[-(x + iy) + (t + z)\lambda_1\right] & \frac{-i}{\lambda_2}\left[-(x + iy) + (t + z)\lambda_2\right] \end{pmatrix} \tag{3.51}$$

These structure coordinates are valid over the whole Shilov boundary space because $\lambda_1 \neq \lambda_2$ everywhere, while the corresponding surface of CP^3, defined by the Kerr polynomial (3.50) is singular.

3.3 Induced metrics on spacetimes

The characteristic property of the present model is that it depends on the integrable complex structure $J_\mu{}^\nu$ and not on a metric $g_{\mu\nu}$. The above approach to the solution of the complex structure equations through the $G_{2,2}$ mathematical machinery permit us to look at spacetimes from a different point of view. The metric should be seen as a product and not as a "primitive" dynamical variable. In fact we may define more than one metric on the 4-dimensional surface of $G_{2,2}$. The complex structure determines its eigenvectors $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ up to four independent Weyl factors. These vectors permit us to define a symmetric tensor $g_{\mu\nu}$ through formula (2.4). This tensor may be used as a metric of the spacetime. The tetrad is apparently null relative to this metric, which is compatible with the complex structure $J_\mu{}^\nu$. In fact the most general metric we may use is

$$g_{\mu\nu} = \Omega^2 \left[(\ell_\mu n_\nu + n_\mu \ell_\nu) - \omega^2 \ (m_\mu \overline{m}_\nu + \overline{m}_\mu m_\nu) \right] \tag{3.52}$$

where $\Omega(x)$ and $\omega(x)$ are two arbitrary real functions. I want to point out that this metric arbitrariness saves the present model from the scalar solitons which caused the most serious difficulties to the Misner-Wheeler geometrodynamic model. Notice that the spherically symmetric spacetimes (e.g. Schwarzschild) are compatible with the Minkowski metric. Therefore they do not differ from the "vacuum" surface.

The above metric is very useful because it is directly related to the complex structure. In fact it determines the complex structure through the algebraic condition (2.12), and in the "vacuum" surface (Minkowski spacetime) it may take a form which respects the remaining Poincaré symmetry. But it is not the only metric we may define. The rank-2 matrices $X^{mi}(\xi)$ permit us to induce the well known $SU(4)$ and $SU(2,2)$ invariant metrics of $G_{2,2}$ down to the 4-dimensional surface. In the bounded (Dirac representation) coordinate neighborhood the surface is

$$z_{11} = \frac{X^{21}X^{12} - X^{11}X^{22}}{X^{01}X^{12} - X^{11}X^{02}} \quad , \quad z_{12} = \frac{X^{01}X^{22} - X^{21}X^{02}}{X^{01}X^{12} - X^{11}X^{02}}$$

$$z_{21} = \frac{X^{31}X^{12} - X^{11}X^{32}}{X^{01}X^{12} - X^{11}X^{02}} \quad , \quad z_{22} = \frac{X^{01}X^{32} - X^{31}X^{02}}{X^{01}X^{12} - X^{11}X^{02}} \tag{3.53}$$

In this coordinate neighborhood the $SU(4)$ and $SU(2,2)$ invariant metrics of $G_{2,2}$ are

$$ds_\pm^2 = \frac{\partial}{\partial \overline{z_{ij}}} \frac{\partial}{\partial z_{kl}} \ln[\det(I \pm z^\dagger z)] \ d\overline{z_{ij}} \ dz_{kl} \tag{3.54}$$

where the "+" and "-" denote the $SU(4)$ and $SU(2,2)$ invariant metrics respectively. These metrics may easily be transcribed in the unbounded (chiral) coordinate neighborhood. After the direct substitution of $z_{ij}(\xi)$, the induced Euclidean metrics on the 4-dimensional surface may be found. These metrics do not seem to be directly related to the complex structure of the surface.

4 VACUA AND EXCITATION MODES

A physically interesting geometrodynamic model must generate the electromagnetic field and the intermediate vector bosons of the Standard Model from the fundamental equations of the model itself, without introducing anything by hand. In the Misner-Wheeler model, the electromagnetic potential is directly and exactly derived from the Rainich conditions. In fact this derivation was the essential reason behind the assumption of the Rainich conditions, as the fundamental equations of the Misner-Wheeler model[10]. In Quantum Field Theory the vacuum excitation modes are the periodic configurations which diagonalize energy and momentum. In the present context periodicity is understood in compactified Minkowski spacetime $M^{\#}$. Therefore we first have to define energy and momentum, and after to look for the model vacua with vanishing energy and momentum, the excitation modes and the solitons with finite energy.

4.1 Physical energy-momentum

It is well known that in any generally covariant model the translation generators are first class constraints, which must vanish. Therefore energy, momentum and angular momentum cannot be defined using Noether's theorem. The success of the Einstein equations strongly suggests that energy-momentum has to be defined through the Einstein tensor $E^{\mu\nu}$. The direct relation of the Einstein tensor with the classical energy-momentum and angular momentum is also strongly implied by the derivation of the equations of motion in the harmonic coordinate system, imposed by the condition $\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \right) = 0$, using the contracted Bianchi identities $\nabla_\mu E^{\mu\nu} = 0$. But $E^{\mu\nu}$ depends on the metric and it is not directly related to the Poincaré generators of the present model. Therefore for the definition of the energy in the present model we proceed as follows[25].

We first consider the coordinate system imposed by the relation

$$\partial_\mu \left(\sqrt{-g} E^{\mu\nu} \right) = 0 \tag{4.1}$$

We next consider the conserved quantity

$$\mathcal{E}(g^{\mu\nu}) = \int_t \sqrt{-g} E^{\mu 0} dS_\mu \tag{4.2}$$

where the time variable t is chosen such that $\mathcal{E}(g^{\mu\nu}) \geq 0$. This quantity depends on the metric $g^{\mu\nu}$ and it does not characterize the complex structure, therefore it cannot be the energy definition of the configuration. We think that the energy of a complex structure is properly defined by the following minimum

$$\mathrm{E}[J_\mu{}^\nu] = \min_{g_{\mu\nu} \in [J_\mu{}^\nu]} \mathcal{E}(g^{\mu\nu}) \tag{4.3}$$

where the minimum is taken over all the class $[J_\mu{}^\nu]$ of metrics (3.52).

Apparently this conserved quantity depends only on the moduli parameters of the complex structure. In a vacuum sector it vanishes, $\mathrm{E}[J_\mu{}^\nu] = 0$. From

the 2-dimensional solitonic models[5], we know that the minima of the energy characterize the solitons. Assuming that $\mathrm{E}[J_\mu{}^\nu]$ is a smooth function of the moduli parameters, we can always expand it around a minimum.

$$\mathrm{E}[J_\mu{}^\nu] \simeq E + \sum_q \varepsilon_q \,\overline{a_q}\, a_q \tag{4.4}$$

where E and ε_q are positive parameters. These variables and a_q are moduli parameters of the complex structure. E is defined to be the energy of the soliton characterized by the minimum and ε_q are the energies of the excitation modes. In the special metric where the minimum (4.3) occurs, we can define the 4-momentum and the angular momentum

$$\mathrm{P}^\nu = \int_t \sqrt{-g} E^{\mu\nu} dS_\mu$$
$$\mathrm{S}^{\mu\nu} = \int_t \sqrt{-g} E^{\rho\sigma} x^\tau \,\Sigma^{\mu\nu}_{\sigma\tau}\, dS_\mu \tag{4.5}$$

These quantities are conserved in the precise coordinate systems, which satisfy (4.1). But this is not enough to identify them with the Poincaré group generators! Recall that the Poincaré transformations are well defined in the present model. They form a subalgebra of $sl(4, C)$ and a part of the infinite algebra of the complex structure preserving transformations. The relation of these Poincaré group generators with the present conserved quantities is implied by the transformation of the Einstein tensor under the Poincaré transformations. Recall that in the unbounded coordinate neighborhood of $G_{2,2}$ the Poincaré transformations do not mix the Hermitian $x_{A'A}$ and the anti-Hermitian $iy_{A'A}$ parts of the projective coordinates $r_{A'A}$. Therefore we must first fix the coordinates to be the Cartesian coordinate system defined as the real part of the $r^{A'A}$ projective coordinates of the Grassmannian manifold $G_{2,2}$. But in this coordinate system energy-momentum is not exactly conserved. It is approximately conserved in the "weak gravity" limit. Therefore we will consider the modes which diagonalize this "weak gravity" limit of energy-momentum. These modes belong to irreducible representations of the Poincaré transformations, properly defined on the Cartesian coordinate system as $\delta x^\mu = \omega^\mu{}_\nu x^\nu + \varepsilon^\mu$. Then the Quantum Theory relation

$$i[\varepsilon Q_\varepsilon \,,\; E^{\mu\nu}] = \delta_\varepsilon E^{\mu\nu} = E^{\mu\rho}\partial_\rho\varepsilon^\nu + E^{\rho\nu}\partial_\rho\varepsilon^\mu - \varepsilon^\rho\partial_\rho E^{\mu\nu} \tag{4.6}$$

implies that P^ν approximately behaves as a vector and $\mathrm{S}^{\mu\nu}$ as an antisymmetric tensor. P^μ and S_z commute with the corresponding Poincaré generators. Hence the approximative relation (4.4) and the preceding Poincaré group transformations imply the forms

$$\mathrm{P}^\mu \simeq k_0^\mu + \sum_{i,s} \int d^3k \; k^\mu \; a_i^+(\vec{k}, s) \; a_i(\vec{k}, s)$$
$$\mathrm{S}_z \simeq s_z + \sum_{i,s} \int d^3k \; s \; a_i^+(\vec{k}, s) \; a_i(\vec{k}, s) \tag{4.7}$$

where the summation is over the momentum, the spin and the irreducible representations i of the Poincaré group. k_0^μ is the 4-momentum and s_z is the z-component of the spin of the soliton. k^μ is the 4-momentum and s is the z-component of the spin of the excitation modes $a_i(\vec{k}, s)$. In the quantized theory the variables $a_i^+(\vec{k}, s)$ and $a_i(\vec{k}, s)$ become the creation and the annihilation operators of the approximative modes, which diagonalize the 4-momentum. From Quantum Field Theory we know that the second parts of (4.7) are formally generated by the ordinary energy momentum tensors of free quantum fields. They should be bosonic because they represent excitation modes. This procedure permit us to write down the Einstein tensor as the energy-momentum tensor of the excitation modes. Notice that this effective energy-momentum tensor has to contain interactions, because the field excitation modes diagonalize the approximative Einstein tensor. We will refer to this effective Lagrangian again below in relation to the computation of the soliton form factors.

The tetrad vectors $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ are the two real and one complex vector fields, which appear in the action of the present model. The number of the Poincaré representations of the excitation modes may be found looking for the independent variables of the tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ which determines the complex structure. It has $(4 \times 4 =)$ 16 real variables, while four real variables are removed by the extended Weyl symmetry (2.19). Hence we find 12 independent variables. Notice that this is the number of bosonic modes which exist in the Standard Model, which has a real massive vector field (3 variables), a complex massive vector field (2×3 variables), the photon (2 variables) and the scalar Higgs field (1 variable). Apparently these arguments are not conclusive and a more direct calculation is needed.

4.2 Conformal and Poincaré vacua

A direct consequence of the present definition of energy and momentum is that all the complex structures which are compatible with the Minkowski metric have zero energy. They determine vacuum configurations. In the context of the Grassmannian manifold formulation of complex structures we see that only the Shilov (characteristic) boundary of the classical domain or its subsurfaces may be vacua. All the complex structures on the closed $S^1 \times S^3$ are vacuum configurations, which are $SU(2,2)$ symmetric, because the surface is invariant. These vacua will be called conformal vacua, because they break global $SL(4, C)$ symmetry group down to $SU(2,2)$ symmetry.

The complex structures on the open "real axis" of the Shilov boundary break the conformal $SU(2,2)$ symmetry down to the $[Poincaré] \times [dilatation]$ group. Recall that the "real axis" subsurface is characterized by a point of the closed Shilov boundary, which fixes the Cayley transformation (3.23). It is the point of the characteristic boundary in the bounded realization, which is sent to "infinity" in the unbounded realization of the classical domain. A general theorem, valid for all classical domains, states that the automorphic analytic transformations, which preserve a point of the characteristic boundary in the bounded

realization, become linear transformations in the unbounded realization of the classical domain[19]. In the present case of the $SU(2,2)$ classical domain these linear transformations form the $[Poincaré] \times [dilatation]$ group. This argument demonstrates that the "real axis" vacuum surface breaks global $SL(4,C)$ down to the $[Poincaré] \times [dilatation]$ group. One may understand the above theorem looking at the following general form of the Cayley transformation which transforms the upper half-plane realization of the $SU(2,2)$ classical domain onto its bounded realization

$$z = U_0 \left(MrM^\dagger - N^\dagger \right) \left(MrM^\dagger - N \right)^{-1} \tag{4.8}$$

where $\det M \neq 0$, $i(N^\dagger - N)$ is negative definite and U_0 is the point of the Shilov boundary, which is sent to infinity. This clear cut emergence of the Poincaré group, through a symmetry breaking mechanism, makes the present model physically very interesting. Recall that the asymptotic flatness condition generates the BMS group, which does not appear in Particle Physics.

The $SU(2) \times U(1)$ transformation $z' = Uz$ changes the characteristic point U_0 of the Cayley transformation (4.8) to UU_0, while it does not affect the Poincaré transformation. That is, it changes the Minkowski spacetime, while it does not change the explicit form of the Poincaré transformation. This implies that the $SU(2) \times U(1)$ transformation commutes with the Poincaré transformation in the following sense: [First make a U_0 preserving Poincaré transformation and after an "internal" U transformation] =[First make an "internal" U transformation and after a UU_0 preserving Poincaré transformation]. I want to point out that these two subgroups of $SU(2,2)$ do not commute in the ordinary sense. I think that this clear cut emergence of the Poincaré group and the "internal" $SU(2) \times U(1)$ group may have some physical relevance.

The dilatation symmetry is broken by the parameter a of the static Kerr polynomial which will be described in the next section. But this proof will be presented here because of the importance of the Poincaré group in physics.

In the next section we will see that the first soliton family is generated by a quadratic polynomial $A_{mn}Z^mZ^n = 0$ with

$$A_{mn} = \begin{pmatrix} \omega_{AB} & p_A{}^{B'} \\ p^{A'}{}_B & 0 \end{pmatrix} \tag{4.9}$$

where $p^{AA'} = \epsilon^{AB}p_B{}^{A'}$. This form is determined assuming invariance under the Poincaré transformations $\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$, with $B_{11}^\dagger B_{22} = I$, $\det(B_{22}) = 1$ and $B_{11}^\dagger B_{21} + B_{21}^\dagger B_{11} = 0$. If we try to impose dilatation symmetry, which has the form $\begin{pmatrix} e^{-\rho} & 0 \\ 0 & e^\rho \end{pmatrix}$, we find $\omega_{AB} = 0$. Apparently this makes the complex structure trivial. Hence ω_{AB}, which generates the spin of the static soliton, breaks the dilatation symmetry leaving the Poincaré group as the largest symmetry.

5 "LEPTONIC" SOLITONS

Standard Model provides a description of weak and electromagnetic interactions through the classification of the left-handed and right-handed field components in the representations of the $U(2)$ group. The electron, muon and heavy lepton (tau) doublets are trivially repeated without any apparent reason. This is the well known family puzzle. No theoretical explanation exists of this dummy repetition of only three representations of the $U(2)$ group. The appearance of the same representations for the quarks obscures the situation. The extension of the unitary group has not yet provided any experimentally acceptable model. The present solitonic model provides a new way to look for a solution to this problem.

5.1 Three families of solitons

The dynamical variables of the present model are the gauge field $A_{j\mu}(x)$ and the integrable tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ up to the extended Weyl symmetry. Notice that the field equations have the characteristic property to admit pure geometric solutions with $A_{j\mu}(x) = 0$. These are the complex structures defined by the integrable tetrad or equivalently a rank 2 matrix $X^{mi}(x)$. The integrable tetrad permits us to define the symmetric tensor (3.52), relative to which the tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ is null. This symmetric tensor may be used as a metric, which contains a large information from the complex structure. It has been pointed out the essential difference between the Euclidean complex structures and the present Lorentzian ones. In the case of Euclidean complex structures the metric is somehow independent of the complex structure. But the Lorentzian complex structure is essentially algebraically fixed by the metric $g_{\mu\nu}$ because the spinor dyad o^A and ι^A, which determines the integrable tetrad, satisfy the algebraic equation (2.12). They are principal directions of the Weyl spinor Ψ_{ABCD}.

The cornerstone of the soliton theory is the regularity of the solitonic configurations. In the present case this is translated to the regularity of the complex manifold. Therefore the Weyl spinor Ψ_{ABCD} must be regular and (o^A, ι^A) are roots of a homogeneous fourth degree polynomial with regular coefficients. Namely, the complex manifold is a covering space of R^4 with a maximum of four sheets and a minimum of two sheets. That is the solitons of the present model are algebraically classified into three classes (families) according to the number of principal null directions of the Weyl tensor as follows:

- The fourth degree polynomial (2.12) is reduced to the square of a homogeneous second degree polynomial $(\Phi_{AB}\xi^A\xi^B)$. These are the type D spacetimes which admit a regular complex structure. We will call it "type D family" of the model.

- The Weyl tensor has three principal null directions, which are geodetic and shear free. These are type II spacetimes in the Petrov classification.

- In the third class the Weyl tensor has four distinct principal null directions, which must also be geodetic and shear free. These are type I spacetimes.

We should notice the amazing similarities of these three classes with the three families of leptons and quarks indicating a completely different approach to the family problem. In conventional Quantum Field Theoretic models the solution to this problem was searched in the context of large simple groups for Grand Unified Theories and supergroups for recent supersymmetric models. In the present model the proposed solution is topological, based on the Petrov classification of the spacetimes, which is well known in General Relativity. The present model shows for the first time that Quantum Field Theory and General Relativity may be intimately related without Grand Unified Theories, Super-symmetry, Supergravity, Strings and Superstrings. The study of the stationary axisymmetric solitons of type D family in the present section will be in the context of this new point of view.

5.2   Massive complex structures of the 1st family

The charged Kerr metric has been extensively studied. After a mass and charge multipole expansion it was observed[1],[11] that this spacetime had the correct electron gyromagnetic ratio $g = 2$. Notice that this extraordinary result came out in the context of pure General Relativity without any reference to Quantum Mechanics or any other particular assumption. This result triggered many attempts to generate particles in the context of pure General Relativity without apparent phenomenological success.

The knowledge of the Poincaré group permit us to look for stationary (static) axisymmetric solitonic complex structures, which will be interpreted as particles of the model with precise mass and angular momentum. In the case of vanishing gauge field, we may use the general solutions (2.8) to find special solutions. In this case the convenient coordinates are

$$z^0 = u + iU \quad , \quad z^1 = \zeta \quad , \quad z^{\widetilde{0}} = v + iV \quad , \quad z^{\widetilde{1}} = \overline{W} \, \zeta \qquad (5.1)$$

where $u = t - r$, $v = t + r$ and $t \in R$, $r \in R$, $\zeta = e^{i\varphi} \tan \frac{\theta}{2} \in S^2$ are assumed to be the four coordinates of the spacetime surface. Assuming the definitions

$$z^0 = i\frac{X^{21}}{X^{01}} \quad , \quad z^1 = \frac{X^{11}}{X^{01}} \quad , \quad z^{\widetilde{0}} = i\frac{X^{32}}{X^{12}} \quad , \quad z^{\widetilde{1}} = -\frac{X^{02}}{X^{12}} \qquad (5.2)$$

we look for solutions which are stable along $s^\mu = (1, 0, 0, 0)$. That is we look for massive solutions such that

$$\delta X^{mi} = i\epsilon^0 [P_0]^m_n X^{ni} \qquad (5.3)$$

where $P_\mu = -\frac{1}{2}\gamma_\mu(1 + \gamma_5)$. It implies

$$\delta X^{0i} = 0 \qquad , \qquad \delta X^{1i} = 0$$

$$\delta X^{2i} = -i\epsilon^0 X^{0i} \qquad , \qquad \delta X^{3i} = -i\epsilon^0 X^{1i} \qquad (5.4)$$

The above definition of the structure coordinates implies

$$\delta z^0 = \epsilon^0 \quad , \quad \delta z^1 = 0$$

$$\delta z^{\widetilde{0}} = \epsilon^0 \quad , \quad \delta z^{\widetilde{1}} = 0$$

(5.5)

and consequently

$$\delta u = \epsilon^0 \quad , \quad \delta U = 0$$

$$\delta v = \epsilon^0 \quad , \quad \delta V = 0$$

(5.6)

$$\delta \zeta = 0 \quad , \quad \delta W = 0$$

This procedure gives stable (time independent) solutions. The little group relative to the vector s^μ is the $SO(3)$ subgroup of the Lorentz group. Therefore we may look for solutions, which are "eigenstates" of the z-component of the spin. In this case the homogeneous coordinates satisfy the following transformations

$$\delta X^{mi} = i\epsilon^{12}[\Sigma_{12}]^m_n X^{ni}$$

(5.7)

where $\Sigma_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. That is we have

$$\delta X^{0i} = -i\frac{\epsilon^{12}}{2}X^{0i} \quad , \quad \delta X^{1i} = i\frac{\epsilon^{12}}{2}X^{1i}$$

$$\delta X^{2i} = -i\frac{\epsilon^{12}}{2}X^{2i} \quad , \quad \delta X^{3i} = i\frac{\epsilon^{12}}{2}X^{3i}$$

(5.8)

The above definition of the structure coordinates implies

$$\delta z^0 = 0 \quad , \quad \delta z^1 = i\epsilon^{12}z^1$$

$$\delta z^{\widetilde{0}} = 0 \quad , \quad \delta z^{\widetilde{1}} = -i\epsilon^{12}z^{\widetilde{1}}$$

(5.9)

and consequently

$$\delta u = 0 \quad , \quad \delta U = 0$$

$$\delta v = 0 \quad , \quad \delta V = 0$$

(5.10)

$$\delta \zeta = i\epsilon^{12}\zeta \quad , \quad \delta W = 0$$

A general solution of (2.8), which satisfies these symmetries, is given by the relations

$$U = U[z^1\overline{z^1}] \quad , \quad V = V[z^{\widetilde{1}}\overline{z^{\widetilde{1}}}]$$

$$W = W[v - u - i(V + U)]$$

(5.11)

A static complex structure is expected to be determined by a Kerr function $K(X^m)$ globally defined on CP^3. Chow's theorem states that every complex

analytic submanifold of CP^n is an algebraic variety (determined by a polynomial). Hence the present complex structure will be determined by a quadratic polynomial invariant under (5.3) and (5.7), which turns out to be

$$Z^1 Z^2 - Z^0 Z^3 + 2aZ^0 Z^1 = 0 \qquad (5.12)$$

The asymptotic flatness condition (3.37) implies

$$U = -2a\frac{z^1\overline{z^1}}{1+z^1\overline{z^1}} \qquad , \qquad V = 2a\frac{z^{\tilde{1}}\overline{z^{\tilde{1}}}}{1+z^1\overline{z^1}} \qquad (5.13)$$

A quite general solution is found if $W\overline{W} = 1$ $(V + U = 0)$. In this case we have the solution

$$U = -2a\sin^2\tfrac{\theta}{2} \qquad , \qquad V = 2a\sin^2\tfrac{\theta}{2}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.14)$$
$$W = \tfrac{r-ia}{r+ia}e^{-2if(r)}$$

A simple investigation shows that this complex structure is (in different coordinates) the static solution (2.32) found in section II using the Kerr-Schild ansatz. One may easily compute the corresponding tetrad up to their arbitrary factors N_1, N_2 and N_3.

$$\ell = N_1[dt - dr - a\sin^2\theta \ d\varphi]$$

$$n = N_2[dt + (\tfrac{r^2+a^2\cos 2\theta}{r^2+a^2} - 2a\sin^2\theta \ \tfrac{df}{dr})dr - a\sin^2\theta \ d\varphi] \qquad (5.15)$$

$$m = N_3[-ia\sin\theta \ (dt - dr) + (r^2 + a^2\cos^2\theta)d\theta + i(r^2 + a^2)\sin\theta d\varphi]$$

The corresponding projective coordinates are

$$r_{0'0} = i\tfrac{X^{21}X^{12}-X^{11}X^{22}}{X^{01}X^{12}-X^{11}X^{02}} = \tfrac{z^0+(z^{\tilde{0}}-ib)z^1 z^{\tilde{1}}}{1+z^1 z^{\tilde{1}}}$$

$$r_{0'1} = i\tfrac{X^{01}X^{22}-X^{21}X^{02}}{X^{01}X^{12}-X^{11}X^{02}} = \tfrac{(z^0-z^{\tilde{0}}+ib)z^{\tilde{1}}}{1+z^1 z^{\tilde{1}}}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.16)$$
$$r_{1'0} = i\tfrac{X^{31}X^{12}-X^{11}X^{32}}{X^{01}X^{12}-X^{11}X^{02}} = \tfrac{(z^0-z^{\tilde{0}}+ib)z^1}{1+z^1 z^{\tilde{1}}}$$

$$r_{1'1} = i\tfrac{X^{01}X^{32}-X^{31}X^{02}}{X^{01}X^{12}-X^{11}X^{02}} = \tfrac{z^{\tilde{0}}+(z^0+ib)z^1 z^{\tilde{1}}}{1+z^1 z^{\tilde{1}}}$$

If these projective coordinates become a Hermitian matrix $x_{A'A}$, then the complex structure is compatible with the Minkowski metric. Otherwise, it is a curved spacetime complex structure. The form (5.14) has been chosen such that for $f(r) = 0$ the complex structure becomes compatible with the Minkowski metric.

The soliton form factor $f(r)$ is expected to be fixed by Quantum Theory, but we have not yet found the precise procedure. We think that any attempt

to give some physical relevance of the present model may come through the identification of the effective energy-momentum tensor of the excitation modes with the bosonic part of the Standard Model energy-momentum tensor. In this case the form factors $f(r)$ of the solitons may be fixed, assuming the condition that the solitons are particle-like sources of the excitation modes. Then the massive static soliton (2.32) with spin $S_z = ma = \frac{h}{2}$, should be identified with the electron. Then it will have (in natural gravitational units $c = G = 1$) mass $m = 6.8 \times 10^{-59} cm$, $a = 1.9 \times 10^{-11} cm$ and charge $q = 1.4 \times 10^{-34} cm$. The complex conjugate complex structure would be the positron.

5.3 Solitonic features of the massive structures

In ordinary Lorentzian Quantum Field Theory the vacuum is determined as the stable state with the lowest energy. Solitons are stable states with finite energies relative to the vacuum. Their configurations are not smoothly deformable to vacuum configurations. We have already revealed the existence of two sets of vacua. The conformally invariant vacua, which are complex structures defined on the closed Shilov boundary $U(2)$ and the Poincaré vacua which are complex structures defined on the open "real axis" of the unbounded neighborhood. In order to reveal the solitons of the model, we have to use the periodicity criteria.

Recall that the ϕ^4-model admits two vacua with $\phi = \pm \frac{\mu}{\sqrt{\lambda}}$. It is well known that the vacuum configurations are periodic, while the soliton configurations are not periodic. This characteristic difference will be used in the present model. The kink configuration and its excitations satisfy the boundary conditions $\phi_{kink}(\pm\infty, t) = \pm\frac{\mu}{\sqrt{\lambda}}$ and the antikink configuration the opposite ones. The corresponding energy-momentum charges are related to the gap of the limit values of the field $\phi(x)$ at $\pm\infty$.

In the present model the excitation modes and the solitons are 4-dimensional surfaces of $G_{2,2}$ which admit integrable tangent vectors in pairs (ℓ, m) and $(n, \overline{m})$. Their essential difference will be on the periodicity of the complex structures they admit. The vacuum surfaces will admit periodic complex structures, while the solitonic complex structures are not periodic on the corresponding surfaces. Therefore we have to specify the precise compactification of the Minkowski spacetime.

Minkowski spacetime is the Poincaré vacuum of the model and it has already been identified with a precise open surface of $G_{2,2}$. It is the "real axis" in the unbounded realization of the classical domain. In the bounded realization of the classical domain, it is an open part of the characteristic (Shilov) boundary. It is precisely limited by the "diagonals" $\tau + \rho = \pi$, $(-\pi \leq \tau - \rho \leq \pi)$, which is $\mathfrak{I}^+$, and $\tau - \rho = \pi$, $(-\pi \leq \tau + \rho \leq \pi)$, which is $\mathfrak{I}^-$. There is an essential difference between Minkowski spacetime and the other asymptotically flat spacetimes. Every null geodesic which originates at some point A^- of $\mathfrak{I}^-$ will pass through the same point A^+ of $\mathfrak{I}^+$. This association permits us to identify A^- with A^+ compactifying Minkowski spacetime[18]. Notice that all the complex structures, which are compatible with the Minkowski metric, are

well defined on compactified Minkowski spacetime $M^{\#}$, because they smoothly cross $\mathfrak{J} = \mathfrak{J}^+ = \mathfrak{J}^-$. The topology of the whole spacetime $M^{\#}$ turns out to be $M^{\#} \sim S^3 \times S^1$.

In order to avoid any misunderstandings, I want to emphasize that there is an essential difference between the present analysis and the corresponding Penrose one. In the present model we deal with complex structures while Penrose deals with Weyl (conformally) equivalent metrics. The present equivalence relation is larger than the Penrose one. A typical example is the Schwarzschild spacetime. It is not Weyl (conformally) equivalent with Minkowski spacetime and it cannot be metrically compactified, because the first derivatives of the metric do not smoothly cross $\mathfrak{J}^+ = \mathfrak{J}^-$. But the complex structure of Schwarzschild spacetime is compatible with Minkowski spacetime because it is trivial. Therefore it can smoothly cross $\mathfrak{J}$ and the Schwarzschild spacetime is a trivial vacuum configuration. In the following example of a static solitonic surface we will consider the Kerr-Schild spacetime but the proof can be extended to any stationary axisymmetric spacetime with $f(-r) \neq f(r)$. It will be shown that the integrable tetrad cannot be smoothly extended across $\mathfrak{J}$. Therefore $\mathfrak{J}^+$ and $\mathfrak{J}^-$ cannot be identified and this complex structure belongs to a soliton sector. The proof of this failure goes as follows:

In order to make things explicit the Kerr-Newman integrable null tetrad will be used as an example. Around $\mathfrak{J}^+$ the coordinates $(u,\ w = \frac{1}{r},\ \theta,\ \varphi)$ are used, where the integrable tetrad takes the form

$$\ell = du - a \sin^2 \theta \ d\varphi$$

$$n = \frac{1 - 2mw + e^2 w^2 + a^2 w^2}{2w^2(1 + a^2 w^2 \cos^2 \theta)}[w^2 \ du - \frac{2(1 + a^2 w^2 \cos^2 \theta)}{1 - 2mw + e^2 w^2 + a^2 w^2} \ dw - aw^2 \sin^2 \theta \ d\varphi]$$

$$m = \frac{1}{\sqrt{2}w(1 + iaw \cos \theta)}[iaw^2 \sin \theta \ du - (1 + a^2 w^2 \cos^2 \theta) \ d\theta -$$
$$-i \sin \theta(1 + aw^2) \ d\varphi]$$

$$(5.17)$$

The physical space is for $w > 0$ and the integrable tetrad is regular on $\mathfrak{J}^+$ up to a factor, which does not affect the congruences, and it can be regularly extended to $w < 0$. Around $\mathfrak{J}^-$ the coordinates $(v,\ w',\ \theta',\ \varphi')$ are used with

$$dv = du + \frac{2(r^2 + a^2)}{r^2 - 2mr + e^2 + a^2} \ dr$$

$$dw' = -dw \quad , \quad d\theta' = d\theta \qquad\qquad (5.18)$$

$$d\varphi' = d\varphi + \frac{2a}{r^2 - 2mr + e^2 + a^2} \ dr$$

and the integrable tetrad takes the form

$$\ell = \tfrac{1}{w'^2}[w'^2 \ dv - \tfrac{2(1+a^2 w'^2 \cos^2 \theta)}{1+2mw'+e^2 w'^2+a^2 w'^2} \ dw' - aw'^2 \sin^2 \theta' \ d\varphi']$$

$$n = \tfrac{1+2mw'+e^2 w'^2+a^2 w'^2}{2(1+a^2 w'^2 \cos^2 \theta')}[dv - a \sin^2 \theta' \ d\varphi']$$

$$m = \tfrac{-1}{\sqrt{2}w'(1-iaw' \cos \theta')}[iaw'^2 \sin \theta \ dv - (1 + a^2 w'^2 \cos^2 \theta') \ d\theta' - \\ -i \sin \theta'(1 + aw'^2) \ d\varphi']$$

(5.19)

The physical space is for $w < 0$ and the integrable tetrad is regular on $\mathfrak{I}^-$ up to a factor, which does not affect the congruences, and it can be regularly extended to $w > 0$. If the mass term vanishes the two regions $\mathfrak{I}^+$ and $\mathfrak{I}^-$ can be identified and the ℓ^μ and n^μ congruences are interchanged, when $\mathfrak{I}^+$ ($\equiv \mathfrak{I}^-$) is crossed. When $m \neq 0$ these two regions cannot be identified and the complex structure cannot be extended across $\mathfrak{I}^+$ and $\mathfrak{I}^-$.

5.4   Hopf invariants of complex structure

We will now consider a classification of the complex structures defined on the $S^1 \times S^3$ surface (the Shilov boundary) of $G_{2,2}$. They are determined by two linearly independent functions $\lambda^{Ai}(\xi)$ in S^2. That is for any complex structure we have two functions

$$S^1 \times S^3 \to S^2 \tag{5.20}$$

It is known that the homotopy group $\pi_1(S^2)$ is trivial but $\pi_3(S^2) = \mathbb{Z}$. The Hopf invariant is determined using the sphere volume 2-form

$$\omega = \frac{i}{2\pi} \frac{d\lambda \wedge d\overline{\lambda}}{(1 + \lambda\overline{\lambda})^2} \tag{5.21}$$

which is closed. This implies that in S^3 there is an exact 1-form ω_1 such that $\omega = d\omega_1$. Then the Hopf invariant of $\lambda(x)$ is

$$H(\lambda) = \int \lambda^*(\omega) \wedge \omega_1 \tag{5.22}$$

In the simple case of a linear polynomial $bZ^0 + Z^2 = 0$, we have

$$\lambda(x) = \frac{t - z + ib}{x - iy} \quad , \quad t = 0 \tag{5.23}$$

The exact 1-form is

$$\omega_1 = \frac{ydx - xdy - bdz}{2\pi(x^2 + y^2 + z^2 + b^2)} \tag{5.24}$$

and

$$H(\lambda) = -\frac{b}{|b|} \tag{5.25}$$

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where we have integrated over the two Minkowski charts, which cover S^3 by simply permitting $r \in (-\infty , +\infty)$. Notice that the present spinor $\lambda^A(\xi)$ is a solution of the linear Kerr polynomial in the unbounded realization and its Hopf invariant is its helicity. In the bounded realization the function which defines the mapping $S^3 \rightarrow S^2$ is different and its Hopf invariant will be different. A simple transformation shows that in the present case the corresponding mapping has zero Hopf invariant.

In the case of the solutions of the quadratic Kerr polynomial

$$\lambda_\pm(x) = \frac{-z + ia \pm \sqrt{x^2 + y^2 + z^2 - a^2 - 2iaz}}{x - iy} \tag{5.26}$$

the Hopf invariant can be computed using its relation to the linking coefficient of two curves in S^3 determined by the inverse images $\lambda^{-1}(\lambda_1) = \{x_1^i(\rho_1)\}$ and $\lambda^{-1}(\lambda_2) = \{x_2^i(\rho_2)\}$. Two general curves are determined using the Lindquist coordinates $(\rho,\ \theta,\ \varphi)$

$$x^i = (\sin\theta\cos\varphi ,\ \sin\theta\sin\varphi ,\ \cos\theta)\rho + a(\sin\varphi ,\ -\cos\varphi ,\ 0) \tag{5.27}$$

for two different values of θ, φ and the variable $\rho \in (-\infty, +\infty)$ in order to cover the whole sphere. Then we know that

$$H(\lambda) = 2\frac{1}{4\pi} \int \frac{\varepsilon_{ijk}(x_1^i - x_2^i)dx_1^j dx_2^k}{|\overrightarrow{x_1} - \overrightarrow{x_2}|^3} \tag{5.28}$$

The two curves can be smoothly deformed to the values $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, $\varphi_2 = 0$. Then the integral becomes

$$H(\lambda_\pm) = \frac{a}{2\pi} \int \int \frac{d\rho_1 d\rho_2}{(\rho_1^2 + \rho_2^2 + a^2)^{\frac{3}{2}}} = \pm\frac{a}{|a|} \tag{5.29}$$

The curved complex structures which are smooth deformations of conformally flat spacetimes will have the same Hopf invariants. This is apparently the case of all the complex structures derived using the Kerr-Schild ansatz. The curved complex structures have Hopf invariant $\frac{a}{|a|}$ at $\mathfrak{J}^+$ and Hopf invariant $-\frac{a}{|a|}$ at $\mathfrak{J}^-$.

5.5   Massless complex structures of the 1st family

We will now consider configurations X^{mi} which are covariant along a null vector $s^\mu = (1, 0, 0, 1)$. Then X^{mi} satisfy the relations

$$\delta X^{mi} = i\frac{\epsilon}{2}[P_0 + P_3]_n^m X^{ni} \tag{5.30}$$

which imply

$$\delta X^{0i} = 0 \quad , \quad \delta X^{1i} = 0$$

$$\delta X^{2i} = 0 \quad , \quad \delta X^{3i} = -i\epsilon X^{1i} \tag{5.31}$$

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In this case the most general quadratic polynomial, which is invariant under the above transformations is

$$(bZ^0 + Z^2)Z^1 = 0 \tag{5.32}$$

Notice that this polynomial determines a singular surface in CP^3, which may be considered as the limit of the corresponding massive Kerr polynomial

$$A_{mn}Z^m Z^n = 0$$

$$A_{mn} = \begin{pmatrix} 0 & 2s & 0 & -E+p \\ 2s & 0 & E+p & 0 \\ 0 & E+p & 0 & 0 \\ -E+p & 0 & 0 & 0 \end{pmatrix} \tag{5.33}$$

with energy E and momentum p in the z-direction in the case of vanishing mass.

The two solutions are $X^{11} = 0$ and $X^{02} = -bX^{22}$. In this case we cannot use the (5.2) definitions of the structure coordinates. Instead we may use the following structure coordinates

$$z^0 = i\frac{X^{21}}{X^{01}} \quad , \quad z^1 = -i\frac{X^{31}}{X^{01}} \quad , \quad z^{\widetilde{0}} = i\frac{X^{32}}{X^{12}} \quad , \quad z^{\widetilde{1}} = \frac{X^{02}}{X^{12}} \tag{5.34}$$

Then they transform as follows

$$\delta z^0 = 0 \quad , \quad \delta z^1 = 0$$
$$\delta z^{\widetilde{0}} = \epsilon \quad , \quad \delta z^{\widetilde{1}} = 0 \tag{5.35}$$

and consequently

$$\delta u = 0 \quad , \quad \delta U = 0$$
$$\delta v = \epsilon \quad , \quad \delta V = 0 \tag{5.36}$$
$$\delta \zeta = 0 \quad , \quad \delta W = 0$$

This procedure gives stable solutions along the null vector s^μ. In the present case the little group is the $E(2)$-like group with the third generator being the same as the previously studied massive case with the $SO(3)$ little group.

The axial symmetry condition (5.7) gives the following infinitesimal transformations

$$\delta X^{0i} = -i\frac{\epsilon^{12}}{2}X^{0i} \quad , \quad \delta X^{1i} = i\frac{\epsilon^{12}}{2}X^{1i}$$
$$\delta X^{2i} = -i\frac{\epsilon^{12}}{2}X^{2i} \quad , \quad \delta X^{3i} = i\frac{\epsilon^{12}}{2}X^{3i} \tag{5.37}$$

The above new definition of the structure coordinates implies

$$\delta z^0 = 0 \quad , \quad \delta z^1 = i\epsilon^{12}z^1$$
$$\delta z^{\widetilde{0}} = 0 \quad , \quad \delta z^{\widetilde{1}} = -i\epsilon^{12}z^{\widetilde{1}} \tag{5.38}$$

Using the same coordinates u, v, ζ a general complex structure solution is

$$U = U[u,\ z^1\overline{z^1}] \quad , \quad V = V[z^{\tilde{1}}\overline{z^{\tilde{1}}}] \quad , \quad W = W[u + iU] \tag{5.39}$$

Notice that these relations are not interconnected and they define a general solution without additional conditions.

In the case of the invariant Kerr quadratic polynomial (5.32) we have $X^{11} = 0$ and $X^{22} = -bX^{02}$. Then the asymptotic flatness conditions (3.37) imply

$$U = 0 \quad , \quad V = bz^{\tilde{1}}\overline{z^{\tilde{1}}} \quad , \quad W = W[u] \tag{5.40}$$

5.6   The 2nd and 3rd family solitons may be unstable

It has already been pointed out that the integrability condition $\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0$ classifies the complex structures into those with 2, 3, and 4 algebraic sheets. The complex structure with 2 sheets is the type D family and it has been extensively studied in the previous subsections. The application of the same procedure to the type II and type I families will show that they may not have stable (static) configurations. That is we will look for an eigenconfiguration which will be invariant under time translation and z-rotation, and we will find that they do not exist.

The starting point is the reasonable assumption that for an asymptotically flat static spacetime there is a coordinate system such that the Weyl tensor has to approach a Penrose twistor, that is $\Psi_{ABCD} \simeq f\omega_{ABCD}$. This means that the Kerr function can locally become equivalent to a quartic polynomial $K(Z) = A_{mnpq}Z^mZ^nZ^pZ^q = 0$. The same result is found applying Chow's theorem. The existence of an axially symmetric configuration implies the existence of quartic polynomials such that

$$\delta K(Z) = C\varepsilon^{12}K(Z) \tag{5.41}$$

This relation can be easily solved. The following five solutions are found:

1. The first solution has $C = 2i$ and it contains only the components Z^0 and Z^2. The polynomial $K(Z)$ takes the form

$$\begin{aligned} A_{0000}(Z^0)^4 + A_{0002}(Z^0)^3(Z^2) + A_{0022}(Z^0)^2(Z^2)^2 + \\ + A_{0222}(Z^0)(Z^2)^3 + A_{2222}(Z^2)^4 = 0 \end{aligned} \tag{5.42}$$

   which defines a singular surface in CP^3.

2. Another solution has $C = -2i$ and it is singular too because it contains only the components Z^1 and Z^3.

3. The third solution has $C = i$ and the polynomial $K(Z)$ takes the form

$$\begin{aligned} A_{0001}(Z^0)^3(Z^1) + A_{0003}(Z^0)^3(Z^3) + A_{0012}(Z^0)^2(Z^1)(Z^2) + \\ + A_{0023}(Z^0)^2(Z^2)(Z^3) + A_{0122}(Z^0)(Z^1)(Z^2)^2 + \\ + A_{0223}(Z^0)(Z^2)^2(Z^3) + A_{1222}(Z^1)(Z^2)^3 + A_{2223}(Z^2)^3(Z^3) = 0 \end{aligned} \tag{5.43}$$

4. The fourth solution has $C = -i$ and $K(Z)$ has a form analogous to the above with $Z^0 \Leftrightarrow Z^1$ and $Z^2 \Leftrightarrow Z^3$ interchanged.

5. The final solution has $C = 0$ and $K(Z)$ takes the form

$$
\begin{aligned}
&A_{0011}(Z^0)^2(Z^1)^2 + A_{0013}(Z^0)^2(Z^1)(Z^3) + A_{0033}(Z^0)^2(Z^3)^2+ \\
&+A_{0112}(Z^0)(Z^1)^2(Z^2) + A_{0123}(Z^0)(Z^1)(Z^2)(Z^3)+ \\
&+A_{0233}(Z^0)(Z^2)(Z^3)^2 + A_{1122}(Z^1)^2(Z^2)^2+ \\
&+A_{1223}(Z^1)(Z^2)^2(Z^3) + A_{2233}(Z^2)^2(Z^3)^2 = 0
\end{aligned}
\tag{5.44}
$$

The stability condition relative to time translation

$$
\delta K(Z) = C\varepsilon^0 K(Z)
\tag{5.45}
$$

can now be applied on the above axially symmetric Kerr polynomials. I find that the only regular (in CP^3) surface comes from the fifth case. The invariant polynomial is

$$
A(Z^1 Z^2 - Z^0 Z^3)^2 + B(Z^0 Z^1)(Z^1 Z^2 - Z^0 Z^3) + C(Z^0 Z^1)^2 = 0
\tag{5.46}
$$

which may be written as the product of two quadratic polynomials which give the type D complex structures. Hence we may conclude that the complex structures (particles) with 3 and 4 sheets cannot be stable.

6 "HADRONIC" SOLITONS AND CONFINEMENT

Quark confinement is actually based on the $SU(3)$ gauge group and the non proven yet hypothesis that the non-Abelian gauge field interactions produce a confining potential. The perturbative potential of the ordinary Yang-Mills action is Coulomp-like $\frac{1}{r}$. The ordinary Yang-Mills action also generates the strong P (CP) problem, because it admits instantons which permit tunnelling between the gauge vacua. The real vacuum of the model is a θ-vacuum which generates a parity violation topological term in the action. The axion particle solution of this problem is expected to be tested in a LHC experiment. The present model trivially solves these problems because its modified Yang-Mills action generates a linear static potential and it does not have instantons.

The amazing similarity between the quark flavor parameters and the leptons is also a puzzle. The quarks look like leptons with a "color". The theoretical efforts to solve this quark-lepton correspondence in the context of Grand Unified Theories have affronted serious problems with the cosmological proton decay bounds. The present model provides a different way to approach this problem. It seems to imply that in some approximation for each "leptonic" (pure geometric) soliton there should be a gauge field excited soliton, which must be perturbatively confined because of the linear static potential.

The variables of the present model are the gauge field $A_{j\mu}(x)$ and the integrable tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$. Notice that the field equations have the characteristic property to admit pure geometric solutions. These solutions may be replaced back into the new covariant Yang-Mills equations and find the corresponding gauge field solutions. The simple solution $A_{j\mu}(x) = 0$ corresponds to the pure geometric "leptonic" solitons without any gauge field interaction.

In complete analogy to the 2-dimensional kinks, we may quantize around the soliton complex structure[25]. Then the gauge field configurations have an asymptotically linear potential instead of the Coulombian ($\frac{1}{r}$). This is a clear indication that these excitation modes cannot exist free and they must be confined into "colorless" bound states which remind us the hadrons. These bound states will be hadronic-like solitons with non-vanishing gauge field strength which in some approximation look like bound states of the simple "leptonic" excitations through a linear potential. That is in the present model picture the quarks could be gauge field excitations of the leptons and they are perturbatively confined. This very simple picture could explain the complete correspondence between leptons and quarks. Apparently in the present context, Standard Model should be considered as an effective theory, like the phonon Lagrangians in solids and fluids[28]. In order to support the above picture of the soliton sectors, we will first compute the classical potential implied by the present action.

In spherical coordinates (t, r, θ, φ) and in the trivial (vacuum) null tetrad

$$
\begin{aligned}
\ell^\mu &= (1, 1, 0, 0) \\
n^\mu &= \tfrac{1}{2}(1, -1, 0, 0) \\
m^\mu &= \tfrac{1}{r\sqrt{2}}\left(0, 0, 1, \tfrac{i}{\sin\theta}\right)
\end{aligned}
\tag{6.1}
$$

the dynamical variable of the gauge field is $(r \, \sin\theta \, m^\mu A_{j\mu})$. Assuming the convenient gauge condition

$$\overline{m}^\nu \partial_\nu \left(r \, \sin\theta \, m^\mu A_{j\mu}\right) + m^\nu \partial_\nu \left(r \, \sin\theta \, m^\mu A_{j\mu}\right) = 0 \qquad (6.2)$$

the field equation takes the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\left(r \, \sin\theta \, m^\mu A_{j\mu}\right) = [source] \qquad (6.3)$$

The dynamical variable apparently gives a linear classical (time-independent) potential. The other two variables $\ell^\mu A_{j\mu}$ and $n^\mu A_{j\mu}$ of the gauge field decouple and vanish.

Exactly the same approach can be followed in the static soliton sector. The dynamical variable is now

$$A = ((r + ia\cos\theta) \, \sin\theta \, m^\mu A_{j\mu}) \qquad (6.4)$$

and it satisfies the gauge condition

$$(r - ia\cos\theta) \, \overline{m}^\nu \partial_\nu A + (r + ia\cos\theta) \, m^\nu \partial_\nu \overline{A} = 0 \qquad (6.5)$$

The corresponding linear part of the field equation is more complicated but in the asymptotic limit coincides with (6.3). The other variables of the gauge field are r-independent and decouple.

The emergence of the asymptotically linear classical potential implies that the gauge field modes cannot exist free. They must be confined. The gauge field excitations of the pure geometric solitons will also be confined because of the linear potential. An $SU(N)$ gauge group implies that in some approximation there should be N gauge field excitation modes. These states could look like the three colored quarks. That is, N must equal three and the gauge group becomes $SU(2)$. Notice that this mechanism implies the existing in nature correspondence between "leptons" and "quarks". Namely, for each pure geometric soliton there must be three "colored" structures which cannot exist free because of their linear interaction. The confining potential imposes that that the gauge field excitations cannot exist free. But the existence of "colorless" solitons with non vanishing gauge field gauge field configurations has to be proved. These solitons are expected to be described by complicated configurations of the tetrad and gauge field configurations, which satisfy the complicated field equations of the present action.

We will now show that the Euclidean form of the present Yang-Mills action does not admit finite action solutions which are called instantons and measure the tunnelling between the gauge vacua. The proof is based on the fact that in the Euclidean manifolds, the complex structure becomes the ordinary real one with $z^{\widetilde{\alpha}} = \overline{z^\alpha}$. Then the structure coordinate form (2.17) of the action becomes

$$I_G = 2 \int d^4z \, F_{j01} \overline{F_{j01}}$$

$$F_{j01} = \partial_0 A_{j1} - \partial_1 A_{j0} - \gamma \, f_{jik} A_{i0} A_{k1} \qquad (6.6)$$

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which is invariant under a complex gauge transformation $A'_{j\alpha} = A_{j\alpha} + \partial_\alpha \Lambda_j + \gamma f_{jik} \Lambda_i A_{k\alpha}$, where Λ_j are now N complex functions. Assuming the enlarged gauge condition $A_{j0} = 0$ the field equations become

$$\partial_{\bar{0}} F_{j01} = 0$$

$$\partial_{\bar{1}} F_{j01} - \gamma\, f_{jik} \overline{A_{i0}} F_{k01} = 0$$

(6.7)

We see that F_{j01} is an holomorphic function of z^0. On the other hand the finite action solutions must satisfy the condition

$$F_{j01} \overline{F_{j01}} \underset{|z| \to \infty}{\Longrightarrow} 0$$

(6.8)

That is F_{j01} must be bounded as a function of z^0. But we know that the constant is the only bounded holomorphic function. Hence the finite action solutions must have $F_{j01} = 0$. Therefore the present model does not have instantons.

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